The Blow-up Lemma and growing degrees

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RSA, May 2011

(joint work with Julia Böttcher, Yoshi Kohayakawa & Anusch Taraz)

Definition

Let $\varepsilon, \delta > 0$. The graph $(A \dot\cup B, E)$ with $|A| = |B| = n$ is an (ε, δ)-*super-regular* pair if

- $|d(X, Y) - d(A, B)| \leq \varepsilon$ for all $X \subseteq A$, $Y \subseteq B$ with $|X|, |Y| \geq \varepsilon n$,
- $\deg(v) \geq \delta n$ for all $v \in A \cup B$.

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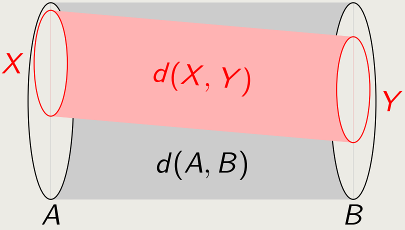

- regularity $\leftrightarrow$ "densities as expected"
- high minimal degree

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The Blow-up Lemma

KOMLÓS, SÁRKÖZY, SZEMERÉDI, '97

For $\delta > 0$, $\Delta \in \mathbb{N}$ there is $\varepsilon > 0$ such that the following holds. If H has $\Delta(H) \leq \Delta$ and $H \subseteq K_{n,n}$ then H is subgraph of any (ε, δ)-super-regular pair $(A \dot\cup B, E)$ with $|A| = |B| = n$.

another proof was given by: RÖDL, RUCIŃSKI, '99

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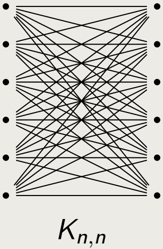

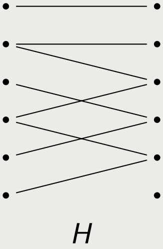

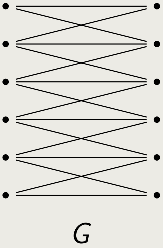

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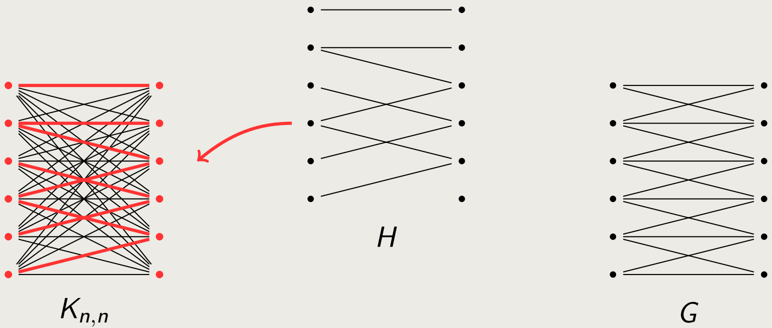

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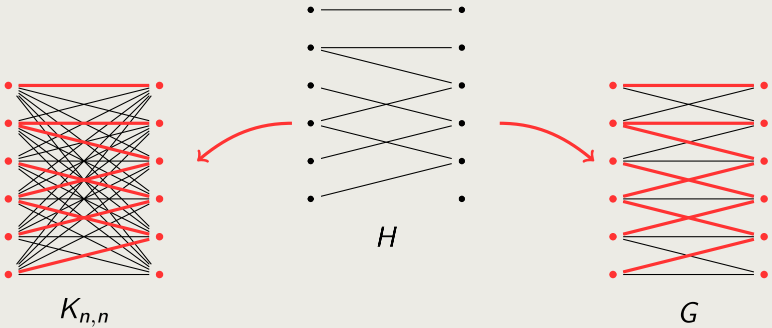

$$H \subseteq K_{n,n} \quad \Rightarrow \quad H \subseteq G$$

Applications of the Blow-up Lemma

Theorems — conjectured by

Graphs with sufficiently high minimal degree contain

- r-th powers of Hamiltonian cycles — Pósa, Seymour, '74
- spanning trees of constant max. degree — Bollobás, '78
- H-factors — Alon, Yuster '96
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Only spanning subgraphs with constant maximal degree!

Spanning subgraphs of growing degrees

Theorem — Komlós, Sárközy, Szemerédi, '95

Graphs with sufficiently high minimal degree contain

- every tree T of order n with $\Delta(T) \leq cn/\log n$.

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Graphs with sufficiently high minimal degree contain

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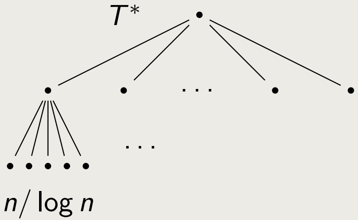
- best possible because $T^* \not\subseteq G_{n,0.9}$



The Blow-up Lemma for growing degrees

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Goal — Böttcher, Kohayakawa, Taraz, W. '11

For $\delta > 0$, $a \in \mathbb{N}$ there is $\varepsilon > 0$ such that the following holds. If H has $\Delta(H) \leq \sqrt{n}/\log n$ and $H \subseteq K_{n,n}$ is a-arrangeable then H is subgraph of any (ε, δ)-super-regular pair $(A \dot\cup B, E)$ with $|A| = |B| = n$.

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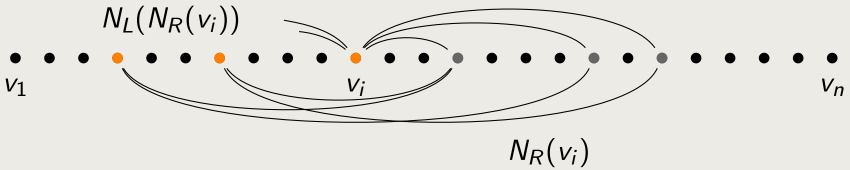
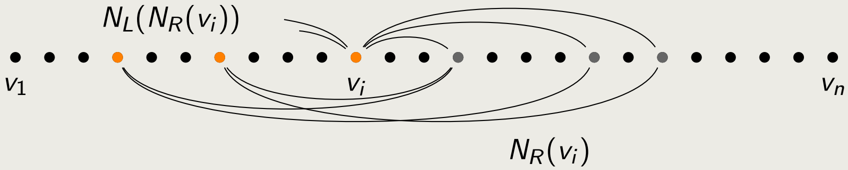
$G = (V, E)$ is *a-arrangeable* if there is an ordering $V = \{v_1, \ldots, v_n\}$ with $|N_L(N_R(v_i))| \leq a$ for all $i = 1, \ldots, n$.

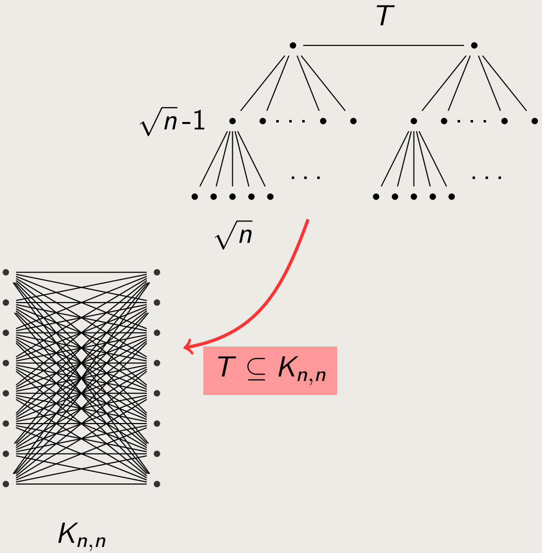


Examples of arrangeable graphs



- trees are 1-arrangeable
- planar graphs are 761-arrangeable — Chen, Shelp '93
- planar graphs are 10-arrangeable — Kierstead, Trotter '94
- graphs without a K_p-subdivision are p^8-arrangeable — Rödl, Thomas '94

Why $\Delta(H) \leq \sqrt{n}$?



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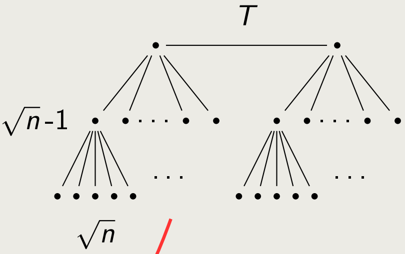

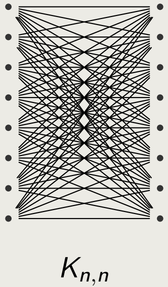

$T \subseteq K_{n,n}$

for every $\varepsilon > 0$ there is an $(\varepsilon, 1/2)$-regular pair $G = (A \dot{\cup} B, E)$ such that $T \not\subseteq G$

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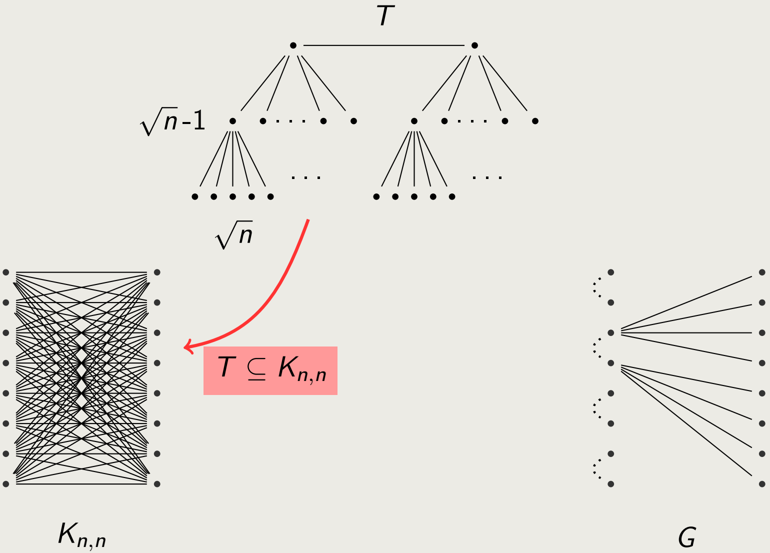

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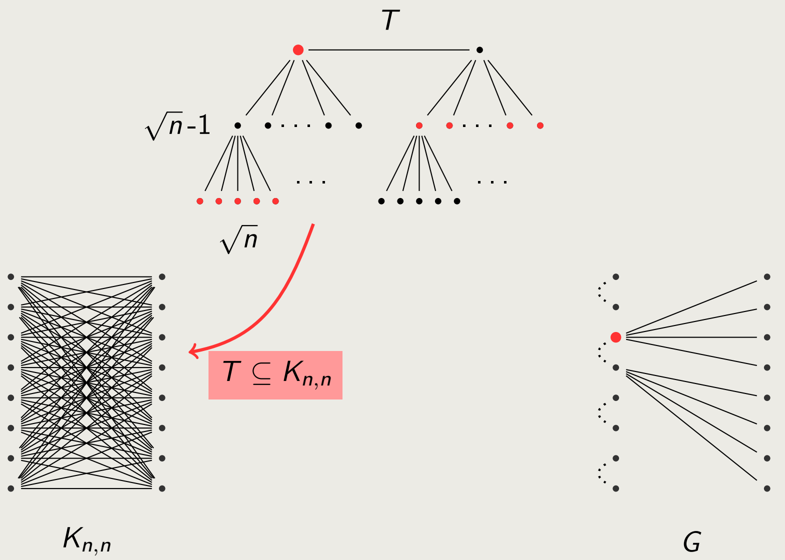

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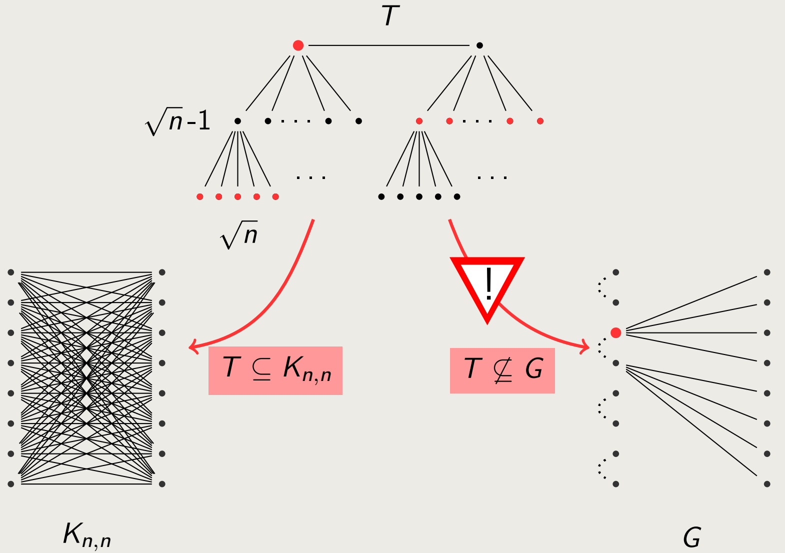

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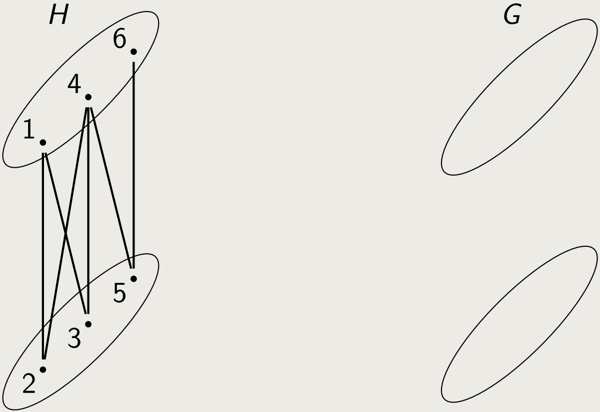
Sketch of proof: a random greedy embedding



(ε, δ)-super-regular

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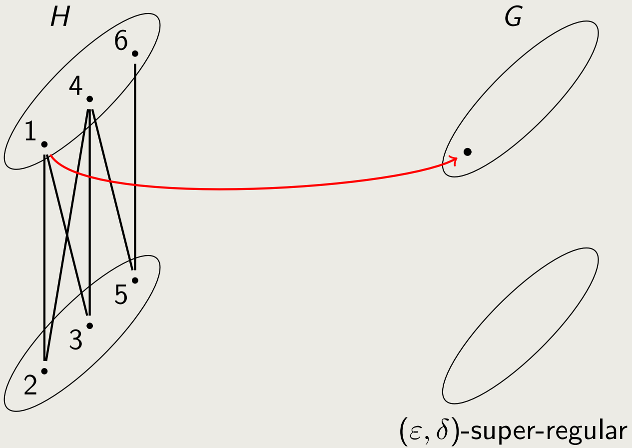

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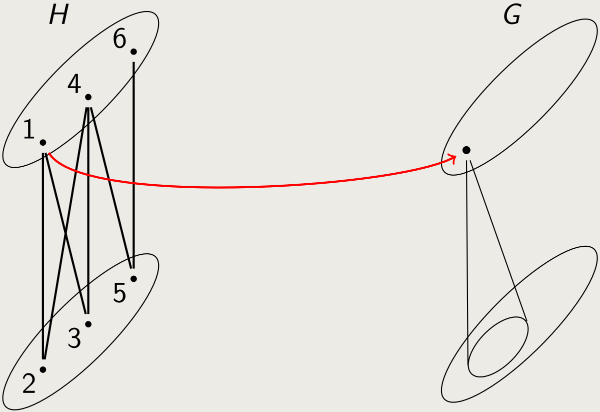

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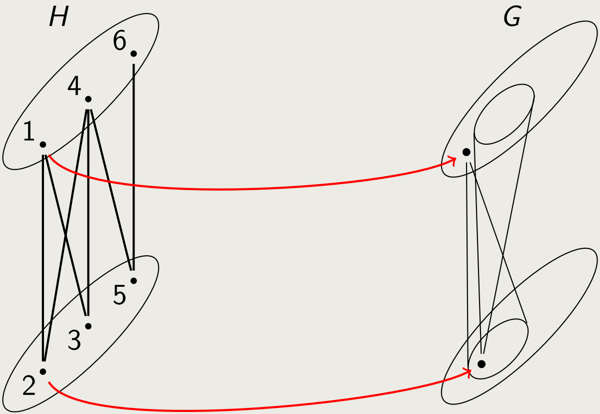

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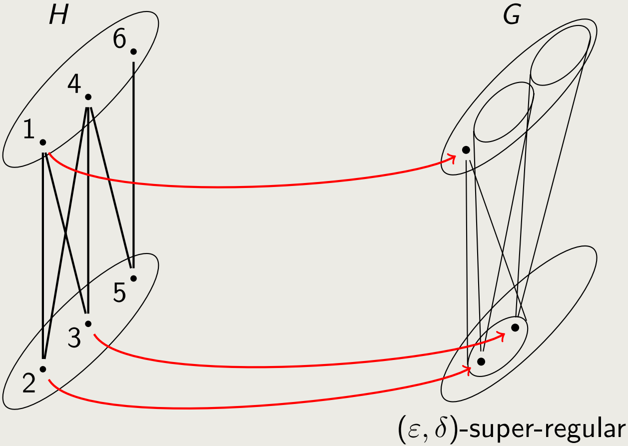

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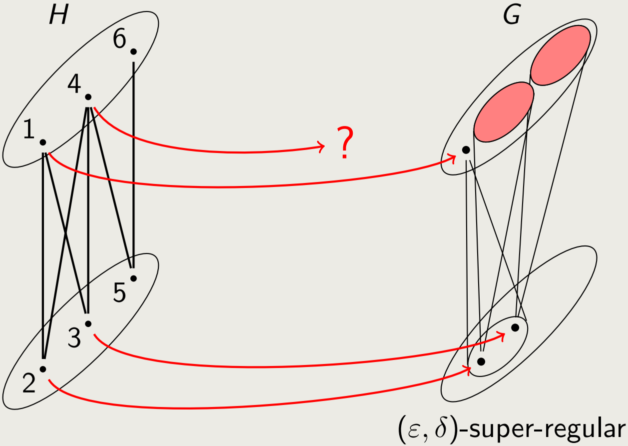

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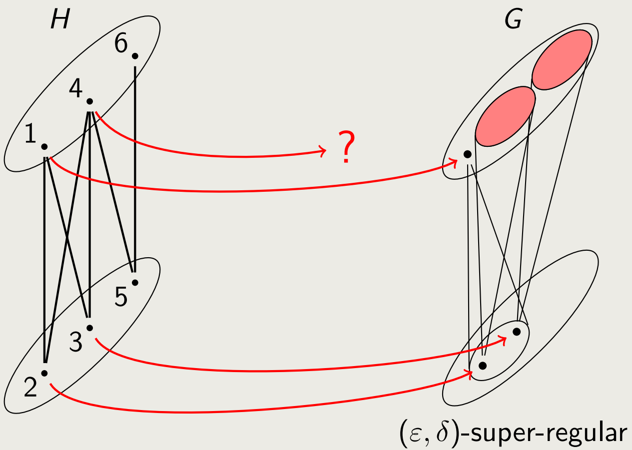

MESSAGE: respect your successors!

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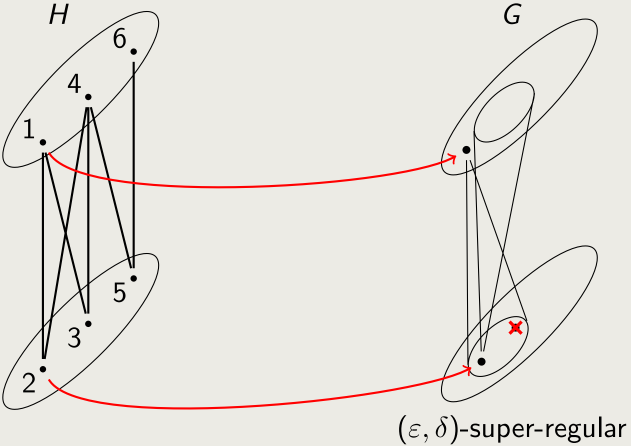

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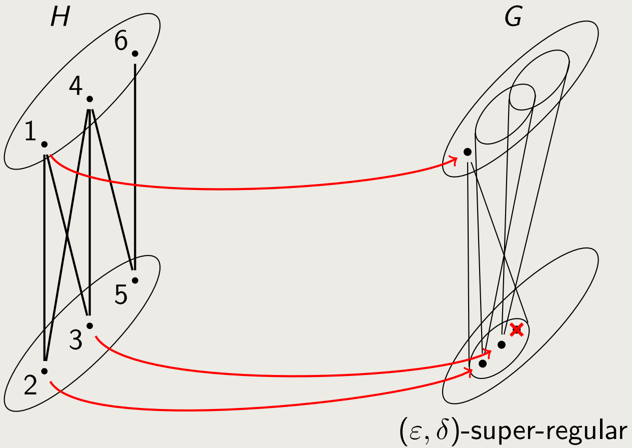

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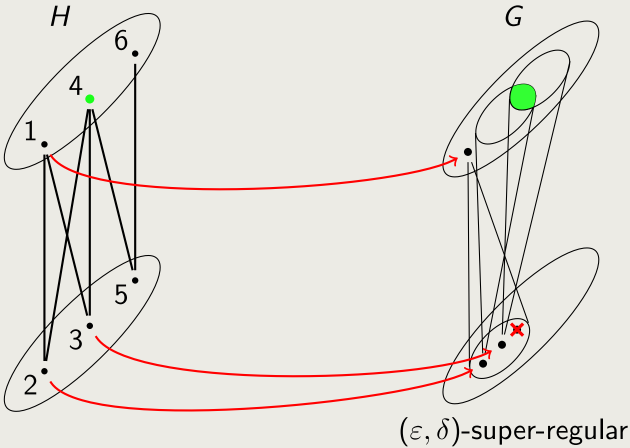

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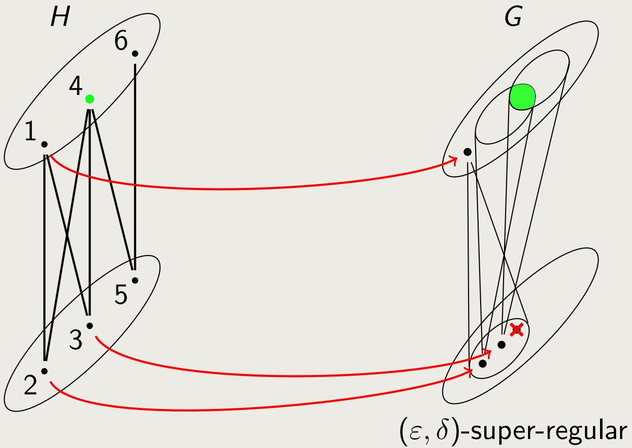

Does this always work?

Respect your successors…

…even if their number is growing with n

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Problem: each successor might *"kill"* a small linear number of candidates

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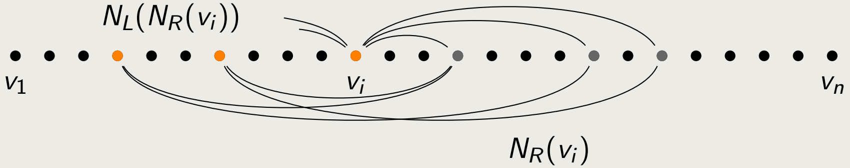

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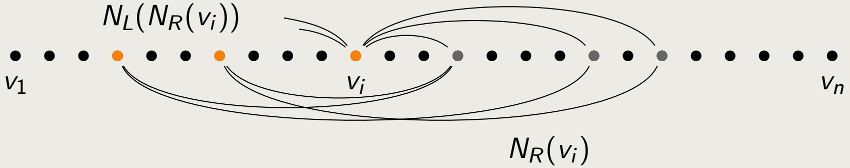
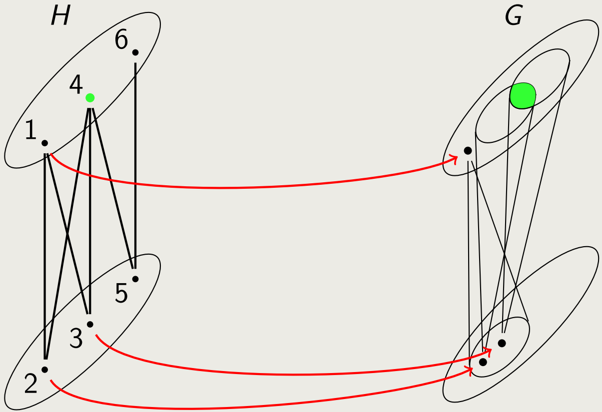


all successors of v_i have at most a predecessors in total

$\Rightarrow$ we have to respect at most 2^a *different* candidate sets

The auxiliary graphs



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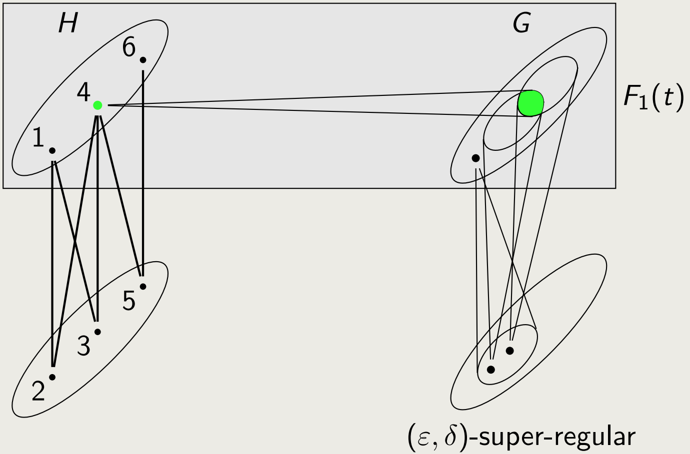

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All auxiliary graphs have high minimal degree with probability going to 1.

Therefore:

There is a perfect matching in the remaining part of each auxiliary graph that completes the **spanning embedding**.

User's wishlist

- landing zones
- variable cluster sizes
- a sparse version
- what else?