EMBEDDING INTO BIPARTITE GRAPHS*

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Abstract. The conjecture of Bollobás and Komlós, recently proved by Böttcher, Schacht, and Taraz [Math. Ann. 343(1), 175–205, 2009], implies that for any $\gamma > 0$, every balanced bipartite graph on $2n$ vertices with bounded degree and sublinear bandwidth appears as a subgraph of any $2n$-vertex graph G with minimum degree $(1+\gamma)n$, provided that n is sufficiently large. We show that this threshold can be cut in half to an essentially best-possible minimum degree of $(\frac{1}{2}+\gamma)n$ when we have the additional structural information of the host graph G being balanced bipartite.

This complements results of Zhao [SIAM J. Discrete Math. 23 (2009), no. 2, 888-900], as well as Hladký and Schacht [SIAM J. Discrete Math. 24 (2010), no. 2, 357-362], who determined a corresponding minimum degree threshold for $K_{r,s}$-factors, with r and s fixed. Moreover, our result can be used to prove that in every balanced bipartite graph G on $2n$ vertices with minimum degree $(\frac{1}{2}+\gamma)n$ and n sufficiently large, the set of Hamilton cycles of G is a generating system for its cycle space.

Key words. Graph theory, Extremal combinatorics, Graph embedding

AMS subject classifications. 05Cxx, 05Dxx

1. Introduction. The Bollobás–Komlós conjecture, recently proved in [11], provides a sufficient and essentially best-possible minimum degree condition for the existence of r-chromatic spanning graphs H of bounded maximum degree and small bandwidth.

A graph is said to have *bandwidth at most b* if there exists an ordering $\{v_1, \dots, v_n\}$ of the vertices, such that for every edge $\{v_i, v_j\}$ of the graph we have $|i-j| \le b$. (For theorems on how the class of n-vertex graphs with $o(n)$ bandwidth relates to other important classes of graphs, see [9].)

THEOREM 1.1 (Böttcher, Schacht, Taraz [11]). *For all $r, \Delta \in \mathbb{N}$ and $\gamma > 0$, there exist constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$ the following holds. If H is an r-chromatic graph on n vertices with $\Delta(H) \le \Delta$ and bandwidth at most βn and if G is a graph on n vertices with minimum degree $\delta(G) \ge (\frac{r-1}{r}+\gamma)n$, then G contains a copy of H.* □

This theorem in particular implies that for any $\gamma > 0$, every bipartite graph H on $2n$ vertices with bounded degree and sublinear bandwidth appears as a subgraph of any $2n$-vertex graph G with minimum degree $(1+\gamma)n$, provided that n is sufficiently large. This bound is essentially best possible for an almost trivial reason: there are graphs G on $2n$ vertices with minimum degree just slightly below n that are not connected and therefore do not contain a connected H as a subgraph.

These counterexamples, however, involve a host graph which is structurally very different from the desired subgraph in the sense that the chromatic number of G is $\Omega(n)$ whereas H is bipartite. One should ask, thus, whether it is possible to lower the minimum degree threshold in Theorem 1.1 for graphs G that are from the outset assumed to have the same chromatic number as H.

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In this paper we answer this question for the case of balanced bipartite graphs, i.e., bipartite graphs on $2n$ vertices with n vertices in each colour class.

Our result can be put into historical context as follows. While Dirac's theorem [17] says that an arbitrary $2n$-vertex graph G with minimum degree at least n contains a Hamilton cycle, it follows as a special case of a theorem of Moon and Moser that in the case of G being balanced bipartite, this minimum degree threshold can be cut almost in half.

Theorem 1.2 (Moon, Moser [41]). *Let G be a balanced bipartite graph on $2n$ vertices. If $\delta(G) \geq \frac{1}{2}n + 1$, then G contains a Hamilton cycle.*

It was subsequently shown by Czygrinow and Kierstead in [16] (for sufficiently large graphs) that the *same* minimum degree bound as in Theorem 1.2 implies the existence not only of a Hamilton cycle but of a bipartite *non*-cyclic n-ladder (interestingly, it seems to be not easy to deduce, at this minimum degree, the existence of a *cyclic* spanning ladder from the existence of a non-cyclic one; it is an open problem whether $\delta(G) \geq \frac{1}{2}n + 1$ in a balanced bipartite graph implies the existence of a cyclic spanning ladder).

In [21] it was proved that slightly increasing the bound $\delta(G) \geq \frac{1}{2}n + 1$ to $\delta(G) \geq (\frac{1}{2} + \gamma)n$ does indeed imply the existence of a *cyclic* spanning ladder. In the present paper we prove that this slightly increased minimum degree bound in fact suffices to obtain *all* balanced bipartite graphs with *bounded maximum degree and sublinear bandwidth* as subgraphs (this e.g. includes all planar bipartite graphs with bounded maximum degree).

Theorem 1.3. *For all γ and Δ there is a positive constant β and an integer n_0 such that for all $n \geq n_0$ the following holds. Let G and H be balanced bipartite graphs on $2n$ vertices such that G has minimum degree $\delta(G) \geq (\frac{1}{2} + \gamma)n$ and H has maximum degree at most Δ and bandwidth at most βn. Then G contains a copy of H.*

We remark that the bandwidth condition in Theorem 1.3 cannot be omitted in the following sense. Abbasi proved in [1] that the assertion of Theorem 1.1 becomes false if $\beta > 4\gamma$ and it is not difficult to see that Abbasi's example can also be used to show that Theorem 1.3 becomes false when, roughly, $\beta > 8\gamma$. The (non-bipartite) host graph which Abbasi uses for his counterexample contains a balanced bipartite graph G meeting our condition on $\delta(G)$ and of course not containing Abbasi's H either. However, the bound on β coming from our proof is very small, having a tower-type dependence on $1/\gamma$.

Related work. In the last two decades, a wealth of results concerning spanning subgraphs in dense graphs have been obtained. In particular, there also seems to be increased interest in the topic of spanning subgraphs in r-partite graphs. This will be corroborated by the following table in which we have collected relevant results concerning spanning subgraphs in host graphs defined by a minimum degree condition. We have sorted the results in the table according to two independent criteria. First, whether the subgraph whose existence is proved consists of subgraphs that are vertex-disjoint copies of a fixed graph F (which we call F-factors) or whether it is a globally connected spanning subgraph. Second, whether the only assumption about the host graph is a high minimum degree, or whether there is an additional assumption about the chromatic number of the host graph. We exclude related topics, such as Ramsey-type results, decomposition results, or results for directed graphs (see [34] for an extensive survey).

Organisation. The proof of Theorem 1.1 relies on the regularity lemma and a

	no restriction on $\chi(G)$	$\chi(G) \leq r$
$H = F$-factor	[45], [5], [3], [18], [29], [27], [12], [28], [35]	[24], [38], [15], [22], [39], [40], [46]
connected H	[17], [6], [42], [2], [31], [32], [43], [33], [13], [7], [26], [11], [25]	[41], [19], [16], [14], this paper¶

Table 1.1
Results (ordered by publication date) on spanning subgraphs in host graphs defined by a minimum-degree condition.

complementing embedding lemma, which we introduce in Section 2. The two main lemmas, an outline of our technique, and the actual proof of Theorem 1.1 are then given in Section 3. The subsequent Sections 4 and 5 are devoted to the proofs of the two main lemmas. We close our paper with the Section 6 which contains remarks on an application of our main result and on a possible generalization of our threshold to an r-partite setting.

2. The regularity method. In this section we formulate a version of Szemerédi's regularity lemma [44] that is convenient for our application (Lemma 2.2), introduce all necessary definitions, and formulate an embedding lemma for spanning subgraphs (Lemma 2.4).

The regularity lemma relies on the concept of a regular pair. To define this, let $G = (V, E)$ be a graph and $0 \leq \varepsilon, d \leq 1$. For disjoint nonempty vertex sets $U, W \subseteq V$ the *density* $d(U, W)$ of the pair (U, W) is the number of edges that run between U and W divided by $|U||W|$. A pair (U, W) with density at least d is (ε, d)*-regular* if $|d(U', W') - d(U, W)| \leq \varepsilon$ for all $U' \subseteq U$ and $W' \subseteq W$ with $|U'| \geq \varepsilon|U|$ and $|W'| \geq \varepsilon|W|$. The following useful property of regular pairs follows immediately from the definition.

Proposition 2.1. *Let $G = (A, B)$ be an (ε, d)-regular pair. Let B' be a subset of B with $|B'| \geq \varepsilon|B|$. Then there are at most $\varepsilon|A|$ vertices in A with less than $(d-\varepsilon)|B'|$ neighbours in B'.* □

The regularity lemma asserts that each graph admits a partition into a constant number (depending only on the desired quality of the partition, not on the graph) of vertex classes of equal size such that most pairs of these classes form an ε-regular pair. The following definition makes this precise. A partition $V_0 \dot\cup V_1 \dot\cup \cdots \dot\cup V_k$ of V with $|V_0| \leq \varepsilon|V|$ is (ε, d)*-regular on* a graph $R = ([k], E_R)$ if $ij \in E_R$ implies that (V_i, V_j) is an (ε, d)-regular pair in G. If such a partition exists, we also say that R is an (ε, d)*-reduced graph* of G. Moreover, R is the *maximal* (ε, d)-reduced graph of the partition $V_0 \dot\cup V_1 \dot\cup \cdots \dot\cup V_k$ if there is no $ij \notin E_R$ with $i, j \in [k]$ such that (V_i, V_j) is (ε, d)-regular. A partition $V_0 \dot\cup V_1 \dot\cup \cdots \dot\cup V_k$ of V is an *equipartition* if $|V_i| = |V_j|$ for all $i, j \in [k]$.

The partition classes V_i with $i \in [k]$ are also called *clusters* of G and V_0 is the *exceptional set*. When the exceptional set V_0 is empty (or when we want to ignore it as well as its size) then we may omit it and say that $V_1 \dot\cup \cdots \dot\cup V_k$ is regular on R. An (ε, d)-regular pair (U, W) is (ε, d)*-super-regular* if every vertex $u \in U$ has degree $\deg_W(u) \geq d|W|$ and every $w \in W$ has $\deg_U(w) \geq d|U|$. For a graph $G = (V, E)$ a partition $V = V_0 \dot\cup V_1 \dot\cup \cdots \dot\cup V_k$ is said to be *super-regular on a graph* R with vertex set

¶Note that all five papers deal with the case $r = 2$.

V_R, $V_R \subseteq [k]$, if (V_i, V_j) is super-regular whenever ij is an edge of R.

In this paper we consider bipartite graphs. The regular partitions that appear in the proof of Theorem 1.3 refine some bipartition and hence their reduced graphs are bipartite as well. More precisely, for a bipartite graph $G = (A \dot\cup B, E)$ we will obtain a partition $(A_0 \dot\cup B_0) \dot\cup A_1 \dot\cup B_1 \dot\cup \cdots \dot\cup A_k \dot\cup B_k$ that is (ε, d)-regular (or super-regular) on some bipartite graph R such that $A = A_0 \dot\cup \cdots \dot\cup A_k$ and $B = B_0 \dot\cup \cdots \dot\cup B_k$. In particular we have two different exceptional sets now, one in A called A_0 and one in B called B_0, each of size εn at most. Such a partition is an equipartition if $|A_1| = |B_1| = |A_2| = \cdots = |A_k| = |B_k|$. In addition, we consider only regular pairs running between the bipartition classes, i.e., pairs of the form (A_i, B_j). Consequently, all reduced graphs (also the maximal reduced graph of a partition) are bipartite.

We now state the version of the regularity lemma that we will use. This is a corollary of the degree form of the regularity lemma and is tailored for embedding applications in balanced bipartite graphs satisfying some minimum degree condition. We sketch its proof below.

Lemma 2.2 (regular partitions of bipartite graphs). *For every $\varepsilon' > 0$ and for every $\Delta, k_0 \in \mathbb{N}$ there exists $K_0 = K_0(\varepsilon', k_0) \in \mathbb{N}$ such that for every $0 \le d' \le 1$, for*

$$\varepsilon'' := \frac{2\Delta\varepsilon'}{1 - \varepsilon'\Delta} \quad \text{and} \quad d'' := d' - 2\varepsilon'\Delta\,,$$

and for every bipartite graph $G = (A \dot\cup B, E)$ with $|A| = |B| \ge K_0$ and $\delta(G) \ge \nu|G|$ for some $0 < \nu < 1$ there exists a graph R and an integer k with $k_0 \le k \le K_0$ with the following properties:

(a) R is an (ε', d')-reduced graph of an equipartition of G and $|V(R)| = 2k$.

(b) $\delta(R) \ge (\nu - d' - \varepsilon'')|R|$.

(c) For every subgraph $R^* \subseteq R$ with $\Delta(R^*) \le \Delta$ there is an equipartition

$$A \dot\cup B = A_0'' \dot\cup B_0'' \dot\cup A_1'' \dot\cup B_1'' \dot\cup \cdots \dot\cup A_k'' \dot\cup B_k''$$

with $A_i'' \subseteq A$ and $B_i'' \subseteq B$ for all $0 \le i \le k$ and (ε'', d'')-reduced graph R, which in addition is (ε'', d'')-super-regular on R^*.

Proof. (sketch) The proof of this lemma is a standard combination of three standard tools. As a first step we simulate the proof of the degree-form (see [31], Lemma 2.1) of the regularity lemma starting with $A \dot\cup B$ as the initial partition. This yields a partition into clusters $A_0, \ldots, B_k$ such that for all vertices $v \notin A_0 \cup B_0$ there are at most $(d' + \varepsilon')n$ edges $e \in E$ with $v \in e$ such that e is not in some (ε', d')-regular pair (A_i, B_j). Hence we get (a). Let R be the maximal (bipartite) (ε', d')-reduced graph of this partition. Then it is easy to see that R inherits the minimum degree condition of G (except for a small loss), see [36, Proposition 9]. This yields (b). Finally, for all pairs (A_i, B_j) with $i, j \in [k]$ that correspond to edges in R^* we take those vertices in A_i or B_j that have too few edges in (A_i, B_j) and move them to A_0 or B_0, respectively. See [36, Proposition 8] for details. This yields (c). □

2.1. Embedding into regular partitions. For embedding *spanning subgraphs* H into graphs G with high minimum degree the blow-up lemma of Komlós, Sárközy and Szemerédi [30] has proved to be an extremely valuable tool.

The blow-up lemma guarantees that bipartite spanning graphs of bounded degree can be embedded into sufficiently super-regular pairs. In fact, this lemma is more general and allows the embedding of graphs H into partitions that are super-regular on some graph R if there is a homomorphism from H to R that does not send too many vertices of H to each cluster of R.

When embedding a spanning graph H into a host graph G, a well-established strategy is to utilise the blow-up lemma on small super-regular "spots" in a regular partition of G for embedding most of the vertices of H, and to use a greedy embedding method to embed the few other vertices first. This embedding method is summarised in the next lemma, the general embedding lemma. Before stating it, we need to identify conditions under which it is possible to proceed in the way just described. This is addressed in the following definition that specifies when a partition of H is "compatible" with a regular partition of G with reduced graph R and a subgraph R' of R such that edges of R' correspond to dense super-regular pairs. In this definition we require that the partition of H has smaller partition classes than the partition of G (condition (i)), and that edges of H run only between partition classes that correspond to a dense regular pair in G (condition (ii)). Further, in each partition class W_i of H we identify two subsets S_i and T_i that are both supposed to be small (condition (iii)). The set S_i contains those vertices that send edges over pairs that do not belong to the super-regular pairs specified by R' and T_i contains neighbours of such vertices.

Definition 2.3 (ε-compatible). *Let $H=(W,E_H)$ and $R=([k],E_R)$ be graphs and let $R'=([k],E_{R'})$ be a subgraph of R. We say that a vertex partition $W=(W_i)_{i\in[k]}$ of H is* ε-compatible *with an integer partition $(n_i)_{i\in[k]}$ of n and with $R'\subseteq R$ if the following holds. For $i\in[k]$ let S_i be the set of vertices in W_i with neighbours in some W_j with $ij\notin E_{R'}$ and $i\neq j$, set $S:=\bigcup S_i$ and $T_i:=N_H(S)\cap(W_i\setminus S)$. Then for all $i,j\in[k]$ we have that*

(i) $|W_i|\le n_i$,
(ii) $xy\in E_H$ *for* $x\in W_i$ *and* $y\in W_j$ *implies* $ij\in E_R$,
(iii) $|S_i|\le\varepsilon n_i$ *and* $|T_i|\le\varepsilon\cdot\min\{n_j : i \text{ and } j \text{ are in the same component of } R'\}$.

The partition $W=(W_i)_{i\in[k]}$ of H is ε-compatible with a partition $V=(V_i)_{i\in[k]}$ of a graph G and with $R'\subseteq R$ if $W=(W_i)_{i\in[k]}$ is ε-compatible with $(|V_i|)_{i\in[k]}$ and with $R'\subseteq R$.

The general embedding lemma asserts that a bounded-degree graph H can be embedded into a graph G if H and G have compatible partitions. A proof can be found in [8, Section 3.3.3].

Lemma 2.4 (general embedding lemma). *For all $d,\Delta,r>0$ there is a constant $\varepsilon=\varepsilon(d,\Delta,r)>0$ such that the following holds.*

Let $G=(V,E)$ be an n-vertex graph that has a partition $V=(V_i)_{i\in[k]}$ with (ε,d)-reduced graph R on $[k]$ which is (ε,d)-super-regular on a graph $R'\subseteq R$ with connected components having at most r vertices each.

Further, let $H=(W,E_H)$ be an n-vertex graph with maximum degree $\Delta(H)\le\Delta$ that has a vertex partition $W=(W_i)_{i\in[k]}$ which is ε-compatible with $V=(V_i)_{i\in[k]}$ and $R'\subseteq R$. Then $H\subseteq G$. □

For applying the general embedding lemma to *spanning* graphs H we would like to have a partition of the graph H whose partition classes match the sizes of a regular partition of G *precisely*. However, usually we cannot guarantee that this is the case for a regular partition obtained from Lemma 2.2. Hence it will become necessary to modify such a regular partition slightly by moving some vertices into different clusters. The following lemma asserts that the resulting partition is still regular with somewhat worse parameters.

For a proof see [10, Proposition 8].

Proposition 2.5. *Let (A,B) be an (ε,d)-regular pair and let $\hat{A}$ and $\hat{B}$ be vertex sets with $|\hat{A}\triangle A|\le\alpha|\hat{A}|$ and $|\hat{B}\triangle B|\le\beta|\hat{B}|$. Then $(\hat{A},\hat{B})$ is an $(\hat{\varepsilon},\hat{d})$-regular pair* *where*

$$\hat{\varepsilon} := \varepsilon + 3(\sqrt{\alpha} + \sqrt{\beta}) \qquad \text{and} \qquad \hat{d} := d - 2(\alpha + \beta).$$

If, moreover, (A, B) is (ε, d)-super-regular and each vertex v in $\hat{A}$ has at least $d|\hat{B}|$ neighbours in $\hat{B}$ and each vertex v in $\hat{B}$ has at least $d|\hat{A}|$ neighbours in $\hat{A}$, then $(\hat{A}, \hat{B})$ is $(\hat{\varepsilon}, \hat{d})$-super-regular with $\hat{\varepsilon}$ and $\hat{d}$ as above. □

3. The proof of the main theorem. In the proof of Theorem 1.3 we will use the general embedding lemma (Lemma 2.4). For applying this lemma we need compatible partitions of the graphs G and H which are provided by the next two lemmas. We start with the lemma for G which constructs a regular partition of G whose reduced graph R contains a perfect matching within a Hamilton cycle of R. The lemma guarantees, moreover, that the regular partition is super-regular on this perfect matching (see Figure 3.1) and that the cluster sizes in the partition can be slightly changed.

We remark that, throughout, $A\dot\cup B$ will denote the vertex set of the host graph G while $X\dot\cup Y$ is the vertex set of the bipartite graph H we would like to embed. The sets A_i and B_i with $i \in [k]$ for some integer k will denote the clusters of a regular partition of G as well as the vertices of a corresponding reduced graph.

Lemma 3.1 (Lemma for G). *For every $\gamma > 0$ there exists $d_{\text{LG}} > 0$ such that for every $\varepsilon > 0$ and every $k_0 \in \mathbb{N}$ there exist $K_0 \in \mathbb{N}$ and $\xi_{\text{LG}} > 0$ with the following properties:*

For every $n \geq K_0$ and for every balanced bipartite graph $G = (A\dot\cup B, E)$ on $2n$ vertices with $\delta(G) \geq \big(1/2 + \gamma\big)n$ there exists $k_0 \leq k \leq K_0$ and a partition $(n_i)_{i\in[k]}$ of n with $n_i \geq n/(2k)$ such that for every partition $(a_i)_{i\in[k]}$ of n and $(b_i)_{i\in[k]}$ of n satisfying $a_i \leq n_i + \xi_{\text{LG}} n$ and $b_i \leq n_i + \xi_{\text{LG}} n$, for all $i \in [k]$, there exist partitions

$$A = A_1\dot\cup \cdots \dot\cup A_k \quad \text{and} \quad B = B_1\dot\cup \cdots \dot\cup B_k$$

such that

(G1) $|A_i| = a_i$ *and* $|B_i| = b_i$ *for all* $i \in [k]$,
(G2) (A_i, B_i) *is* $(\varepsilon, d_{\text{LG}})$*-super-regular for every* $i \in [k]$.
(G3) (A_i, B_{i+1}) *is* $(\varepsilon, d_{\text{LG}})$*-regular for every* $i \in [k]$.

The proof of this lemma is presented in Section 4. The following lemma, which we will prove in Section 5, constructs the corresponding partition of H.

It guarantees that the $2k$ partition classes of H are roughly of the same sizes as the corresponding partition classes of G (see (H3)), and that all edges of H are mapped to edges of a cycle C on $2k$ vertices and all edges except those incident to a very small set S (see (H1)) are in fact mapped to the edges of a perfect matching in C (see (H2)).

Lemma 3.2 (Lemma for H). *For every $k \in \mathbb{N}$ and every $\xi > 0$ there exists $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and for every balanced bipartite graph $H = (X\dot\cup Y, F)$ on $2n$ vertices satisfying* $\text{bw}(H) \leq \beta n$ *and for every integer partition $n = n_1 + \cdots + n_k$ with $n_i \leq n/8$ there exists a set $S \subseteq V(H)$ and a graph homomorphism $f\colon V(H) \to V(C)$, where C is the cycle on the vertices $A_1, B_2, A_2, \ldots, B_k, A_k, B_1, A_1$, such that*

(H1) $|S| \leq \xi \cdot 2k \cdot n$,
(H2) *for every $\{x, y\} \in F$ with $x \in X\setminus S$ and $y \in Y\setminus S$ there is $i \in [k]$ such that $f(x) \in A_i$ and $f(y) \in B_i$,*
(H3) $|f^{-1}(A_i)| < n_i + \xi n$ *and* $|f^{-1}(B_i)| < n_i + \xi n$ *for every* $i \in [k]$.

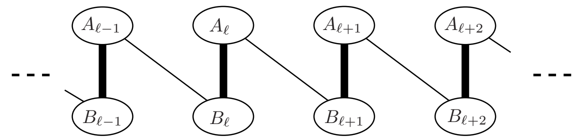

FIG. 3.1. *The regular partition constructed by Lemma 3.1 with regular pairs* (A_i, B_i) *and super-regular pairs* (A_i, B_{i+1}).

With Lemma 2.4 (the general embedding lemma), Lemma 3.1 (the lemma for G) and Lemma 3.2 (the lemma for H) at our disposal, we are ready to give the proof of the main theorem.

Proof. (Proof of Theorem 1.3) Given γ and Δ, let d be the constant provided by Lemma 3.1 for input γ. Let ε be the constant Lemma 2.4 returns for input d, Δ, and $r = 2$. We continue the application of Lemma 3.1 with input ε and $k_0 := 2$ and get constants K_0 and ξ_{LG} and set $\xi_{\text{LH}} := \xi_{\text{LG}}\varepsilon/(100\Delta K_0^2)$. Further, let β be the minimum of all the values β_k and n_0' be the maximum of all the values $n_0^{(k)}$ that Lemma 3.2 returns for input k and ξ, where k runs from k_0 to K_0. Finally, we set $n_0 := \max\{n_0', K_0\}$.

Let $G = (A\dot{\cup}B, E)$ and $H = (X\dot{\cup}Y, F)$ be balanced bipartite graphs on $2n$ vertices with $n \geq n_0$, $\delta(G) \geq (\frac{1}{2} + \gamma)n$, $\Delta(H) \leq \Delta$, and $\text{bw}(H) \leq \beta n$. We apply Lemma 3.1 to the graph G in order to obtain an integer k and an integer partition $(n_i)_{i\in[k]}$ with $n_i \geq \frac{1}{2}n/k$ for all $i \in [k]$. Next, we apply Lemma 3.2 to the graph H and the integer partition $(n_i)_{i\in[k]}$ and get a vertex set $S \subseteq X \cup Y$ and a homomorphism f from H to the cycle C on vertices $A_1, B_2, A_2, \dots, B_k, A_k, B_1, A_1$ such that (H1)–(H3) are satisfied. With this we can define the integer partitions $(a_i)_{i\in[k]}$ and $(b_i)_{i\in[k]}$ required for the continuation of Lemma 3.1: set $a_i := |f^{-1}(A_i)|$ and $b_i := |f^{-1}(B_i)|$ for all $i \in [k]$. By (H3) we have $a_i \leq n_i + \xi_{\text{LH}}n \leq n_i + \xi_{\text{LG}}n$ and $b_i \leq n_i + \xi_{\text{LG}}n$ for all $i \in [k]$. It follows that Lemma 3.1 now gives us vertex partitions $A = (A_i)_{i\in[k]}$ and $B = (B_i)_{i\in[k]}$ for G such that (G1)–(G3) hold. We complement this with vertex partitions $X = (X_i)_{i\in[k]}$ and $Y = (Y_i)_{i\in[k]}$ for H defined by $X_i := f^{-1}(A_i)$ and $Y_i := f^{-1}(B_i)$ and claim that we can use the general embedding lemma (Lemma 2.4) for these vertex partitions of G and H.

Indeed, first observe that (G2) and (G3) imply that the partition $V(G) = (A_i)_{i\in[k]} \dot{\cup} (B_i)_{i\in[k]}$ is (ε, d)-regular on the graph C. Further, by (G3) this partition is (ε, d)-super-regular on the graph R' on the same vertices as C and with edges A_iB_i for all $i \in [k]$. Notice that the components of R' have size $r = 2$. It follows that we can apply Lemma 2.4 if the vertex partition $V(H) = (X_i)_{i\in[k]}\dot{\cup}(Y_i)_{i\in[k]}$ is ε-compatible with the partition $V(G) = (A_i)_{i\in[k]}\dot{\cup}(B_i)_{i\in[k]}$ and with $R' \subseteq C$. To check this first note that by (G1) we have $|A_i| = a_i = |X_i|$ and $|B_i| = b_i = |Y_i|$ for all $i \in [k]$ and thus Property (i) of an ε-compatible partition is satisfied. Since f is a homomorphism from H to C we also immediately get Property (ii) for $(X_i)_{i\in[k]}\dot{\cup}(Y_i)_{i\in[k]}$. In addition, since $|A_i| = a_i \leq n_i + \xi_{\text{LH}}n$ for all $i \in [k]$, we also have $|A_i| \geq n_i - k\xi_{\text{LH}}n \geq \frac{1}{2}n/k - k\xi_{\text{LH}}n \geq \Delta\xi_{\text{LH}}2kn/\varepsilon$ by the choice of ξ_{LH}. This together with (H1) implies that $|S \cap A_i| \leq \xi_{\text{LH}}2kn \leq \varepsilon|A_i|$ and $|N_H(S) \cap A_i| \leq \Delta|S| \leq \Delta\xi_{\text{LH}}2kn \leq \varepsilon|A_j|$ for all $i, j \in [k]$. Similarly we get $|S \cap B_i| \leq \varepsilon|B_i|$ and $|N_H(S) \cap B_i| \leq \varepsilon|B_j|$ for all $i, j \in [k]$. This clearly implies Property (iii) of an ε-compatible partition.

Accordingly we can apply Lemma 2.4 to the graphs G and H with their partitions $V(G) = (A_i)_{i\in[k]}\dot{\cup}(B_i)_{i\in[k]}$ and $V(H) = (X_i)_{i\in[k]}\dot{\cup}(Y_i)_{i\in[k]}$, respectively, which implies that H is a subgraph of G. ∎

4. A regular partition of G with a spanning cycle. In this section we will prove the Lemma for G. This lemma is a consequence of the regularity lemma (Lemma 2.2), Theorem 1.2, and the following lemma which states that, under certain circumstances, we can adjust a (super)-regular partition in order to meet a request for slightly differing cluster sizes.

LEMMA 4.1. *Let $k \geq 1$ be an integer, $0 < \xi \leq 1/(20k^2)$ and let $G = (A\dot{\cup}B, E)$ be a balanced bipartite graph on $2n$ vertices with partitions $A = A_1'\dot{\cup}\cdots\dot{\cup}A_k'$ and $B = B_1'\dot{\cup}\cdots\dot{\cup}B_k'$ such that $|A_i'|, |B_i'| \geq n/(2k)$ and (A_i', B_i') is (ε', d')-super-regular and (A_i', B_{i+1}') is (ε', d')-regular for all $i \in [k]$. Let $(a_i')_{i\in[k]}$ and $(b_i')_{i\in[k]}$ be integers such that $a_i', b_i' \leq \xi n$ for all $i \in [k]$ and $\sum_{i\in[k]} a_i' = \sum_{i\in[k]} b_i' = 0$. Then there are partitions $A = A_1\dot{\cup}\cdots\dot{\cup}A_k$ and $B = B_1\dot{\cup}\cdots\dot{\cup}B_k$ with $|A_i| = |A_i'| + a_i'$ and $|B_i| = |B_i'| + b_i'$ and such that (A_i, B_i) is (ε, d)-super-regular and (A_i, B_{i+1}) is (ε, d)-regular for all $i \in [k]$ where $\varepsilon := \varepsilon' + 100k\sqrt{\xi}$ and $d := d' - 100k^2\sqrt{\xi} - \varepsilon'$.*

Proof. The lemma will be proved by performing a simple redistribution algorithm that will iteratively adjust the cluster sizes. Throughout the process, we denote by A_i and B_i the changing clusters, beginning with $A_i := A_i'$ and $B_i := B_i'$. We call A_i a *sink* when $|A_i| < |A_i'| + a_i'$, and a *source* when $|A_i| > |A_i'| + a_i'$, and analogously for B_i'. Each iteration of the algorithm will have the effect that the number of vertices in a single source decreases by one, the number of vertices in a single sink increases by one, and all other cluster cardinalities stay the same.

We start by describing one iteration of the algorithm. Obviously, as long as not every cluster in A has exactly the desired size, there is at least one source. We choose an arbitrary source A_i, and, as will be further explained below, the regularity of the pair (A_i, B_{i+1}) implies that within A_i there is a large set of vertices each of which can be added to the neighbouring cluster A_{i+1} while preserving the super-regularity of the pair (A_{i+1}, B_{i+1}). We do this with one arbitrary vertex from this set. Thereafter, within A_{i+1} there is again a large set of vertices (the newly arrived vertex may or may not be one of them) suitable for being moved into A_{i+2} while preserving the super-regularity of the pair (A_{i+2}, B_{i+2}), and we again do this with one arbitrary vertex from this set. We then continue in this way until for the first time we move a vertex into a sink. (It may happen that it is not the vertex we initially took out of A_i that arrives in the sink.) This is the end of the iteration.

We repeat such iterations as long as there are sources, i.e. we choose an arbitrary source and repeat what we have just described. Since each iteration ends with adding a vertex to a sink while not changing the cardinality of the clusters visited along the way, we do not increase the number of vertices in any source, let alone create a new source, and hence after a finite number of iterations (which we will estimate below) the algorithm ends with no sources remaining and therefore all clusters within A having exactly the desired size.

We then repeat what we have just described for the clusters within B, the only difference being that vertices get moved from B_i into B_{i-1}, not B_{i+1}, since only in this direction a regular pair can be used ((A_{i-1}, B_i) is regular, (A_{i+1}, B_i) need not be regular).

We now analyse the algorithm quantitatively. Clearly, the total number of iterations (we call it t) is at most the sum of all positive a_i' and all positive b_i'. Obviously, both the sum of all positive a_i' and the sum of all positive b_i' is bounded from above by $\frac{1}{2}k\xi n$, hence

$$t \leq \tfrac{1}{2}k\xi n + \tfrac{1}{2}k\xi n = k\xi n. \tag{4.1}$$

We will now use this bound together with Proposition 2.5 to estimate the effect of the redistribution on the regularity and density parameters. Since in each iteration each cluster receives at most one vertex and loses at most one vertex, for every $i \in [k]$ and after any step of the algorithm, we have

$$|A_i \Delta A_i'| \leq 2t \leq 2k\xi n\,,$$

and analogously $|B_i \Delta B_i'| \leq 2k\xi n$. We now invoke Proposition 2.5 on the pairs (A_i, B_i) and (A_i, B_{i+1}), once with $\hat{A} := A_i$, $\hat{B} := B_i$ once with $\hat{A} := A_i$, $\hat{B} := B_{i+1}$ and we claim that we may use $\alpha := \beta := 16k^2\xi$. Indeed, we have $|A_i| \geq |A_i'| - t \geq n/(2k) - 2k\xi n$ and because $\xi \leq 1/(20k^2)$ implies $2k\xi n \leq 5k\xi n - 20k^3\xi^2 n$, hence $|A_i \Delta A_i'| \leq 2k\xi n \leq (5k\xi - 20k^3\xi^2)n = 10k^2\xi(n/(2k) - 2k\xi n) \leq \alpha|A_i|$, and analogously $|B_i \Delta B_i'| \leq \beta|B_i|$. By Proposition 2.5, every pair (A_i, B_i) and (A_i, B_{i+1}) is $(\hat{\varepsilon}, \hat{d})$-regular with $\hat{\varepsilon} := \varepsilon' + 24k\sqrt{\xi}$ and $\hat{d} := d' - 64k^2\xi$, hence $\hat{\varepsilon} \leq \varepsilon$ and $\hat{d} \geq d$, proving the parameters claimed in the lemma, as far as mere regularity goes.

As for the claimed super-regularity of the vertical pairs, let A_i, B_i and B_{i+1} be clusters at an arbitrary step of the algorithm. Using Proposition 2.1 and (4.1) we know that the pairs (A_i, B_i) and (A_i, B_{i+1}) being $(\hat{\varepsilon}, \hat{d})$-regular implies that there are at least $(1-\hat{\varepsilon})|A_i|$ vertices in A_i having at least $(\hat{d}-\hat{\varepsilon})|B_{i+1}| - t \geq (\hat{d}-\hat{\varepsilon})|B_{i+1}| - 2k\xi n$ neighbours in B_{i+1}, and it remains to prove that $(\hat{d}-\hat{\varepsilon})|B_{i+1}| - 2k\xi n \geq d|B_{i+1}|$ which is equivalent to $2k\xi n/|B_{i+1}| \leq 100k^2\sqrt{\xi} - 64k^2\xi - 24k\xi$. Because of $2k\xi n/|B_{i+1}| \leq 2k\xi n/(|B_{i+1}'| - t) \leq 2k\xi n/(n/2k - 2k\xi n) = 4k^2\xi/(1 - 4k^2\xi)$ it is therefore sufficient that $4k^2\xi/(1 - 4k^2\xi) \leq 100k^2\sqrt{\xi} - 64k^2\xi - 24k\sqrt{\xi}$ and it is easy to check that this is true by the hypothesis on ξ. □

Now we will prove Lemma 3.1. To this end we will apply Lemma 2.2 to the input graph G. By (a) and (b) of Lemma 2.2 we obtain a regular partition with a bipartite reduced graph R of high minimum degree. Theorem 1.2 then guarantees the existence of a Hamilton cycle in R which will imply property (G3). This Hamilton cycle serves as R^* in Lemma 2.2(c) which promises a regular partition of G that is super-regular on R^*. For finishing the proof we will use a greedy strategy for distributing the vertices into the exceptional sets over the clusters of this partition (without destroying the super-regularity required for (G2)) and then apply Lemma 4.1 to adjust the cluster sizes as needed for (G1).

Proof. (Proof of Lemma 3.1) Let $\gamma > 0$ given. We assume without loss of generality that $\gamma < 1/20$ and set $d_{\text{LG}} := \gamma^2/100$. Now let $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ be given. We assume that $\varepsilon \leq \gamma^2/1000$, since otherwise we can set $\varepsilon := \gamma^2/1000$, prove the lemma, and all statements will still hold for any larger ε.

Our next task is to choose ε' and d'. For this, consider the following functions in ε' and d':

$$\begin{aligned} \varepsilon'' &:= \frac{\varepsilon'}{1 - 2\varepsilon'}\,, & \hat{\varepsilon} &:= \varepsilon'' + 6\sqrt{\varepsilon''/\gamma(1-\varepsilon'')}\,, \\ d'' &:= d' - 4\varepsilon'\,, & \hat{d} &:= d'' - 4\varepsilon''/\gamma(1-\varepsilon'')\,. \end{aligned} \tag{4.2}$$

Observe that

$$\varepsilon' \ll \varepsilon'' \ll \hat{\varepsilon} \qquad \text{and} \qquad \hat{d} \ll d'' \ll d'\,,$$

by which we mean, for example, that $\varepsilon' \leq \varepsilon''$ but that we can make ε'' arbitrarily small by choosing ε' sufficiently small. Keeping in mind that $\gamma < 1/20$, it is easy to check that when setting $\varepsilon' := \varepsilon^3\gamma^3$ and $d' := \varepsilon + \gamma^2$, the following inequalities are all satisfied:

$$\hat{\varepsilon} \le \tfrac{1}{10}\varepsilon\,, \qquad \hat{d} - \varepsilon \ge 2d_{\rm LG}\,, \qquad \gamma - d' - \varepsilon'' > 0 \tag{4.3}$$

$$(\tfrac{1}{2} + \gamma - \varepsilon'')(1-d'')^{-1} \ge \tfrac{1}{2} + \tfrac{2}{3}\gamma\,, \qquad d''(1-d'')^{-1} \le \tfrac{1}{6}\gamma\,. \tag{4.4}$$

Next, using (4.3), we can choose an integer k_0' with $k_0 \le k_0'$ such that for all integers k with $k_0' \le k$ we have

$$(\gamma - d' - \varepsilon'')k \ge 1\,. \tag{4.5}$$

Apply Lemma 2.2 with ε', $\Delta := 2$, and with k_0 replaced by k_0', to obtain K_0. Choose $\xi_{\rm LG} > 0$ such that

$$100K_0\sqrt{\xi_{\rm LG}} \le \tfrac{1}{10}\varepsilon, \quad 100(K_0)^2\sqrt{\xi_{\rm LG}} \le d_{\rm LG}. \tag{4.6}$$

Now let G be given. Feed d' and G into Lemma 2.2 and obtain $k \in \mathbb{N}$ with $k_0 \le k_0' \le k \le K_0$ together with an equipartition of G into $2k+2$ classes and an (ε', d')-reduced graph R on $2k$ vertices by (*a*) of Lemma 2.2. By assumption $\delta(G) \ge (\frac{1}{2}+\gamma)n$, so setting $\nu := 1/2 + \gamma$ and making use of part (*b*) of Lemma 2.2, we get

$$\delta(R) \ge (\tfrac{1}{2} + \gamma - d' - \varepsilon'')|V(R)| = \tfrac{1}{2}|V(R)| + (\gamma - d' - \varepsilon'')k \overset{(4.5)}{\ge} \tfrac{1}{2}|V(R)| + 1.$$

We infer from Theorem 1.2 that R contains a Hamilton cycle R^*. Now apply part (*c*) of Lemma 2.2 and obtain an equipartition of G which is (ε'', d'')-regular on R, (ε'', d'')-super-regular on R^*, and has classes

$$A = A_0''\dot\cup\cdots\dot\cup A_k'' \quad \text{and} \quad B = B_0''\dot\cup\cdots\dot\cup B_k''.$$

Obviously, R and thus R^* are bipartite and so, without loss of generality (renumbering the clusters if necessary), we can assume that the Hamilton cycle R^* consists of the vertices representing the classes

$$A_1'', B_2'', A_2'', B_3'', \ldots, B_k'', A_k'', B_1'', A_1''$$

with edges in this order. Therefore, we know that the pairs (A_i'', B_i'') and (A_i'', B_{i+1}'') are (ε'', d'')-super-regular for all $i \in [k]$. Let $L := |A_i''| = |B_i''|$ and observe that

$$(1-\varepsilon'')\frac{n}{k} \le L \le \frac{n}{k}\,.$$

Our next aim is to get rid of the classes A_0'' and B_0'' by moving their vertices to other classes. We will do this, roughly speaking, as follows. When moving a vertex $x \in A_0''$ to some class A_i'', say, we will move an arbitrary vertex $y \in B_0''$ to the corresponding class B_i'' at the same time. We will also make sure that x has at least $d''|B_i''|$ neighbours in B_i'' and y has at least $d''|A_i''|$ neighbours in A_i''. Here are the details for this procedure. For an arbitrary pair $(x,y) \in A_0'' \times B_0''$ we define

$$I(x,y) := \Big\{ i \in [k] \colon \quad |N_G(x) \cap B_i''| \ge d''\,|B_i''| \quad \text{and} \quad |N_G(y) \cap A_i''| \ge d''\,|A_i''| \Big\}\,.$$

We claim that for every $(a,b) \in A_0'' \times B_0''$ we have $|I(x,y)| \ge \gamma k$. To prove this claim, first recall that $L = |A_i''| = |B_i''|$ for all $i \in [k]$. Define

$$I(x) := \big\{ i \in [k] \colon |N_G(x) \cap B_i''| \ge d''|B_i''| \big\}\,,$$
$$I(y) := \big\{ i \in [k] \colon |N_G(y) \cap A_i''| \ge d''|A_i''| \big\}\,.$$

As $|A''_0| = |B''_0| \le \varepsilon'' n$ we have

$$\begin{aligned}(\tfrac{1}{2}+\gamma)n \;\le\; \deg_G(x) \;\le\; &|I(x)|L + (k-|I(x)|)\,d''L + \varepsilon'' n \\ =\; &|I(x)|(1-d'')L + kd''L + \varepsilon'' n\,.\end{aligned}$$

and hence

$$\begin{aligned}|I(x)| &\ge \frac{(\frac{1}{2}+\gamma)n - kd''L - \varepsilon'' n}{(1-d'')L} = \frac{(\frac{1}{2}+\gamma-\varepsilon'')}{1-d''}\frac{n}{L} - \frac{d''}{1-d''}\,k \\ &\overset{(4.4)}{\ge} (\tfrac{1}{2}+\tfrac{2}{3}\gamma)k - \tfrac{1}{6}\gamma k = (\tfrac{1}{2}+\tfrac{1}{2}\gamma)k\,.\end{aligned}$$

Similarly, $|I(y)| \ge (\frac{1}{2}+\frac{1}{2}\gamma)k$. Since $I(x)$ and $I(y)$ are both subsets of $[k]$, this implies that $|I(x,y)| = |I(x) \cap I(y)| \ge \gamma k$, which proves the claim.

We group the vertices in $A''_0 \cup B''_0$ into (at most $\varepsilon'' n$) pairs $(x,y) \in A''_0 \times B''_0$ and choose an index $i \in I(x,y)$ which has the property that (A''_i, B''_i) has so far received a minimal number of additional vertices. Then we move x into A''_i and y into B''_i. Hence, at the end, every cluster A''_i, or B''_i gains at most $\varepsilon'' n/(\gamma k)$ additional vertices. Denote the final partition obtained in this way by

$$A \dot\cup B = \hat A_1 \dot\cup \hat B_1 \dot\cup \ldots \dot\cup \hat A_k \dot\cup \hat B_k\,.$$

Set $\alpha := \beta := \varepsilon''/\gamma(1-\varepsilon'')$ and observe that

$$\frac{\varepsilon'' n}{\gamma k} = \alpha(1-\varepsilon'')\frac{n}{k} \le \alpha L\,.$$

So Proposition 2.5 tells us that for all $i \in [k]$ the pairs $(\hat A_i, \hat B_i)$ are still $(\hat\varepsilon, \hat d)$-super-regular and the pairs $(\hat A_i, \hat B_{i+1})$ are still $(\hat\varepsilon, \hat d)$-regular, because

$$\begin{aligned}&\hat\varepsilon \overset{(4.2)}{=} \varepsilon'' + 6\sqrt{\varepsilon''/\gamma(1-\varepsilon'')} = \varepsilon'' + 3(\sqrt{\alpha}+\sqrt{\beta}) \qquad \text{and} \\ &\hat d \overset{(4.2)}{=} d'' - 4\varepsilon''/\gamma(1-\varepsilon'') = d'' - 4\alpha = d'' - 2(\alpha+\beta)\,.\end{aligned}$$

Now we return to the statement of Lemma 3.1. We set $n_i := |\hat A_i| = |\hat B_i|$ for all $i \in [k]$. Let $(a_i)_{i\in[k]}$ and $(b_i)_{i\in[k]}$ be given and set $a''_i := a_i - n_i$ and $b''_i := b_i - n_i$. Then

$$a''_i \le \xi_{\text{LG}} n, \quad b''_i \le \xi_{\text{LG}} n, \quad \sum_{i\in[k]} a''_i = \sum_{i\in[k]} a_i - \sum_{i\in[k]} n_i = n - n = 0 = \sum_{i\in[k]} b''_i\,.$$

Therefore we can apply Lemma 4.1 with parameter ξ_{LG} to the graph G with partitions $\hat A_1 \dot\cup \cdots \dot\cup \hat A_k$ and $\hat B_1 \dot\cup \cdots \dot\cup \hat B_k$. Since

$$\begin{aligned}\hat\varepsilon + 100k\sqrt{\xi_{\text{LG}}} &\overset{(4.3),(4.6)}{\le} \tfrac{1}{10}\varepsilon + \tfrac{1}{10}\varepsilon \le \varepsilon \qquad \text{and} \\ \hat d - 100k^2\sqrt{\xi_{\text{LG}}} - \varepsilon &\overset{(4.3),(4.6)}{\ge} 2d_{\text{LG}} - d_{\text{LG}} = d_{\text{LG}}\,,\end{aligned}$$

we obtain sets A_i and B_i for each $i \in [k]$ such that $|A_i| = |\hat A_i| + a''_i = n_i + a''_i = a_i$ and $|B_i| = b_i$, and with the property that (A_i, B_i) is (ε, d)-super-regular and (A_i, B_{i+1}) is (ε, d)-regular. This completes the proof of Lemma 3.1. □

5. The proof of the Lemma for H.

5.1. Summary of the proof. In this section we prove the Lemma for H (Lemma 3.2) and in this subsection we summarise the proof. In the beginning, H is cut into small pieces of exactly equal size along a bandwidth ordering (ordering of the vertices of H that respects the bandwidth bound). This makes H controllable in the sense that we have the guarantee that edges of H either run within one piece or from one piece to its successor, but that other edges do not exist. Our goal is to 'weave' this path-like graph onto the (much smaller) Hamilton cycle C within the reduced graph. This cycle *has already been prepared* at this point by the Lemma for G (Lemma 3.1). In particular, the sizes of the clusters A_i and B_i have already been decided upon except that the Lemma for G tolerates small final adjustments of at most $\xi_{\text{LG}}n$ vertices per cluster. The crucial point about the proof of the Lemma for H is not to demand more than the Lemma for G is willing to tolerate.

Importantly, we do not (care to) know anything about how the bandwidth ordering moves back and forth between the bipartition classes of H. Therefore, the equal overall sizes of pieces do not imply equal sizes of pieces *per colour class*—which are the sizes that really count when it comes to putting a particular piece into an edge (i.e. bipartite graph) A_iB_i. This is what thwarts the following naive attempt at weaving H onto C: Distribute the pieces in the order induced by the bandwidth ordering to the edges A_iB_i, without making any 'jumps', and then trust to luck that for each i, both A_i and B_i get filled-up approximately at the same time (so that one can move on to the edge $A_{i+1}B_{i+1}$ without leaving one of A_i or B_i further from being filled-up exactly than what the Lemma for G can tolerate). This, however, need not come to pass at all and does not seem to be easy to guarantee even when one tries to find a bandwidth ordering specially fitted for this purpose (which we do *not* do in the present solution).

Our solution to this problem is to force luck to be on our side by using the probabilistic method to show that there exists *some* way of assigning the pieces of H to the pairs A_iB_i so that all A_i and B_i are approximately filled to a precision within the tolerances of the Lemma for G. In such an assignment, there are typically large 'jumps' from one pair A_iB_i to another $A_{i'}B_{i'}$ with $i' > i$. The details of this preparatory argument are given in Subsection 5.2. At this point, we have only assigned the pieces of H but have not 'woven' anything yet: the edges running from one of the pieces to its successor do not necessarily fit into the reduced graph C, i.e. we do not yet have a homomorphism $V(H) \to B_1, A_1, B_2, A_2, B_3, A_3, B_4, \ldots, B_k, A_k, B_1 = C$. To correct this, we finally resort to a greedy deterministic 'linking' procedure, presented in Subsection 5.3. It robs the approximately filled pieces of a tiny number of vertices whose attached H-edges are then woven all the way from one random piece to the next. In the end we will have succeeded in constructing a homomorphism $V(H) \to C$ and will still have kept the demands for adjustment within the tolerances of the Lemma for G.

5.2. Approximate assignment. Our goal is to group small pieces $W_1, \ldots, W_\ell$ of the balanced bipartite graph H on $2n$ vertices into packages $P_1, \ldots, P_k$ that form balanced bipartite subgraphs of H. This is equivalent to the following problem. Given the sizes a_j and b_j of the colour classes of each piece W_j (i.e., a_j counts the vertices of W_j that are in X and b_j those that are in Y) we know that the a_j's sum up to n and the b_j's sum up to n. Then we would like to have a mapping $\varphi : [\ell] \to [k]$ such that for all $i \in [k]$ the a_j with $j \in \varphi^{-1}(i)$ sum up approximately to the same value as the b_j with $j \in \varphi^{-1}(i)$. The following lemma asserts that such a mapping φ exists. The package P_i will then (in the proof of Lemma 3.2) consist of all pieces W_j with

$j \in \varphi^{-1}(i)$.

LEMMA 5.1. *For all $0 < \xi \le 1/4$ and all positive integers k there exists $\ell \in \mathbb{N}$ such that for all integers $n \ge \ell$ the following holds. Let $(n_i)_{i\in[k]}$, $(a_j)_{j\in[\ell]}$, and $(b_j)_{j\in[\ell]}$ be integer partitions of n such that $n_i \le \frac{1}{8}n$ and $a_j+b_j \le (1+\xi)\frac{2n}{\ell}$ for all $i \in [k]$, $j \in [\ell]$. Then there is a map $\varphi : [\ell] \to [k]$ such that for all $i \in [k]$ and $\bar{a}_i := \sum_{j\in\varphi^{-1}(i)} a_j$ and $\bar{b}_i := \sum_{j\in\varphi^{-1}(i)} b_j$ we have*

$$\bar{a}_i < n_i + \xi n \qquad \textit{and} \qquad \bar{b}_i < n_i + \xi n\,. \tag{5.1}$$

In the proof of Lemma 5.1 we will use a Chernoff bound and the following formulation of a concentration bound due to Hoeffding.

THEOREM 5.2 (Hoeffding bound [4, Theorem A.1.16]). *Let $X_1,\dots,X_s$ be independent random variables with $\mathbb{E}X_i = 0$ and $|X_i| \le 1$ for all $i \in [s]$ and let X be their sum. Then $\mathbb{P}[|X| \ge a] \le 2\exp(-a^2/(2s))$.* □

Proof. (Proof of Lemma 5.1) For the proof of this lemma we use a probabilistic argument and show that under a suitable probability distribution a random map satisfies the desired properties with positive probability.

For this purpose set $\ell := \lceil 1000k^5/\xi^2 \rceil$ and construct a random map $\varphi\colon [\ell] \to [k]$ by choosing $\varphi(j) = i$ with probability n_i/n for $i \in [k]$, independently for each $j \in [\ell]$. To show that this map satisfies (5.1) with positive probability we first estimate the sum of all a_j's and b_j's assigned to a fixed $i \in [k]$. To this end, let $\mathbb{1}_j$ be the indicator variable for the event $\varphi(j) = i$ and define a random variable $S_i := \sum_{j\in[\ell]} \mathbb{1}_j$. Clearly S_i is binomially distributed, we have $\mathbb{E}S_i = \ell\frac{n_i}{n}$, and by the Chernoff bound $\mathbb{P}[|S_i| \ge \mathbb{E}S_i + t] \le 2\exp(-2t^2/\ell)$ (cf. [23, Remark 2.5]) we get

$$\mathbb{P}\Big[\big|S_i - \ell\frac{n_i}{n}\big| \ge \tfrac{1}{2}\xi\ell\Big] \le 2\exp(-\tfrac{1}{2}\xi^2\ell).$$

Next, we examine the difference between the sum of the a_j's assigned to i and the sum of the b_j's assigned to i. We define random variables $D_{i,j} := \frac{\ell}{3n}(a_j - b_j)(\mathbb{1}_j - \frac{n_i}{n})$ and set $D_i := \sum_{j\in[\ell]} D_{i,j}$. Then $\mathbb{E}D_{i,j} = 0$ and as $a_j + b_j \le \frac{3n}{\ell}$ we have $|D_{i,j}| \le 1$. Thus Theorem 5.2 implies

$$\mathbb{P}\big[|D_i| \ge \tfrac{1}{6}\xi\ell\big] \le 2\exp(-\tfrac{1}{72}\xi^2\ell).$$

By the union bound, the probability that we have

$$|S_i - \ell\tfrac{n_i}{n}| < \tfrac{1}{2}\xi\ell \qquad \text{and} \qquad |D_i| < \tfrac{1}{6}\xi\ell \qquad \text{for all } i \in [k] \tag{5.2}$$

is therefore at least $1 - k\cdot 2\exp(-\frac{1}{2}\xi^2\ell) - k\cdot 2\exp(-\frac{1}{72}\xi^2\ell)$ which is strictly greater than 0 by our choice of ℓ. Therefore there exists a map φ with (5.2). We claim that this map satisfies (5.1). To see this, observe first that $\frac{3n}{\ell}D_i = \sum_{j\in\varphi^{-1}(i)}(a_j - b_j) = \bar{a}_i - \bar{b}_i$ which together with (5.2) implies $\bar{a}_i - \bar{b}_i < \frac{1}{2}\xi n$. Moreover, we have $S_i = |\varphi^{-1}(i)|$ and

$$\begin{aligned}\bar{a}_i &= \tfrac{1}{2}(\bar{a}_i + \bar{b}_i) + \tfrac{1}{2}(\bar{a}_i - \bar{b}_i) \le \tfrac{1}{2}(1+\xi)\tfrac{2n}{\ell}|\varphi^{-1}(i)| + \tfrac{1}{2}\cdot\tfrac{1}{2}\xi n \\ &\overset{(5.2)}{<} \frac{1}{2}(1+\xi)\frac{2n}{\ell}\Big(\ell\frac{n_i}{n} + \frac{1}{2}\xi\ell\Big) + \frac{1}{4}\xi n \le n_i + \xi n\end{aligned}$$

where the last inequality follows from $\xi \le \frac{1}{4}$ and $n_i \le \frac{1}{8}n$. Since an entirely analogous calculation shows that $\bar{b}_i < n_i + \xi n$, this completes the proof of (5.1). □

5.3. Linking the random pieces. We will now use Lemma 5.1 to finally prove the Lemma for H (Lemma 3.2) in the manner that has been outlined in the summary at the beginning of the present section.

Proof. (Proof of Lemma 3.2) Let k and ξ be given. Give $\xi' := \xi/4$ and k to Lemma 5.1, get ℓ, set $\beta := \xi'/(4\ell k)$ and $n_0 := \lceil k/\beta \rceil$, and let $H = (X\dot{\cup}Y, F)$ and $(n_i)_{i\in[k]}$ be given as in the statement of the lemma for H.

We assume that the vertices of H are given in a bandwidth ordering, partition $V(H)$ along this ordering into ℓ sets $W_1, \ldots, W_\ell$ of as equal sizes as possible and define $x_i := |W_i \cap X|$ and $y_i := |W_i \cap Y|$. Then $x_i + y_i = |W_i| \le \lceil 2n/\ell \rceil \le 2n/\ell + 1 \le (1+\xi)2n/\ell$ and since $n \ge n_0 \ge \ell$ by definition of n_0 and $n_i \le n/8$ by hypothesis, we can give $(n_i)_{i\in[k]}$, $(x_i)_{i\in[\ell]}$ and $(y_i)_{i\in[\ell]}$ to Lemma 5.1 and get a $\varphi\colon [\ell] \to [k]$ with (5.1). Trivially, we may also get a $\varphi\colon [\ell] \to \{0, 1, \ldots, k-1\}$ with (5.1), and it is this φ that we will use in what follows for the sake of being able to calculate indices in the group $\mathbb{Z}/k\mathbb{Z}$.

We have now arrived at the difficulty already described in the summary in Subsection 5.1. Since the map φ is obtained via the probabilistic method, there is no control over how far apart in the Hamilton cycle C two sets $W_{\varphi(i-1)}$ and $W_{\varphi(i)}$ end up. If there are edges between $W_{\varphi(i-1)}$ and $W_{\varphi(i)}$ we need to guarantee, however, that these edges are mapped to edges of C in order to obtain the desired homomorphism f. To overcome this difficulty, we resort to the aforementioned greedy linking process which robs the pieces W_i of a small number of *linking vertices*, small enough that this modification can still be tolerated by the Lemma for G. The linking vertices are then distributed (always in the 'direction' B_i, A_i, B_{i+1}) over all the clusters lying in-between the cluster pairs $A_{\varphi(i-1)}, B_{\varphi(i-1)}$ and $A_{\varphi(i)}, B_{\varphi(i)}$. This is done in such a way that the edges attached to the linking vertices end up on edges of C. In the process, each A_i and B_i may accumulate many sets of linking pieces, but since these sets are so tiny, it will be possible to declare the still tiny union of all the linking vertices to be the 'special set' S in (H1) (whose role in the proof as a whole is explained by Definition 2.3 and Lemma 2.4 together with the proof of Theorem 1.3).

We now carry out this argument formally. For every $i \in [\ell]$ let w_i be the first vertex in W_i according to the bandwidth ordering fixed at the beginning, capture the length of the 'random jump' by

$$I_i := \left[2 \cdot \big((\varphi(i) - \varphi(i-1)) \mod k\big)\right], \tag{5.3}$$

and define sets of *linking vertices* by setting, for every pair $(i,j) \in [\ell] \times I_i$,

$$L^i_j := [w_i + (j-1)\lceil \beta n \rceil, w_i + j\lceil \beta n \rceil) \subseteq W_i,$$

where adding 1 to a vertex signifies taking its successor in the bandwidth ordering and the half-open interval has its obvious meaning. Finally, abbreviate $L^i := \bigcup_{j\in I_i} L^i_j$ and $NL^i := W_i \backslash L^i$ (the *non-linking vertices*).

Then all L^i_j have the common cardinality $\lceil \beta n \rceil$ and $|L^i| = |I_i| \cdot \lceil \beta n \rceil \le 2(k-1) \cdot \lceil \beta n \rceil \le 2k\beta n$, the latter being implied by $n \ge n_0 \ge k/\beta$. Since $\beta \le 1/(4k\ell)$ implies that $2k\beta n + \beta n \le \lfloor 2n/\ell \rfloor \le |W_i|$ for every $i \in [\ell]$, we have $L^i \subsetneq W_i$ and, for every $i \in [\ell]$, $|NL^i| = |W_i| - |L^i| \ge |W_i| - 2k\beta n \ge (2k\beta n + \beta n) - 2k\beta n = \beta n$, i.e., at the end of every set W_i there definitely are at least βn non-linking vertices (we need this guarantee for our proof but nothing more—actually, the non-linking vertices are vastly in the majority in a 'typical' piece but since this is something we do not use in any way in the formal argument, we do not need to make this more precise). All this is illustrated on the left-hand side of Figure 5.1.

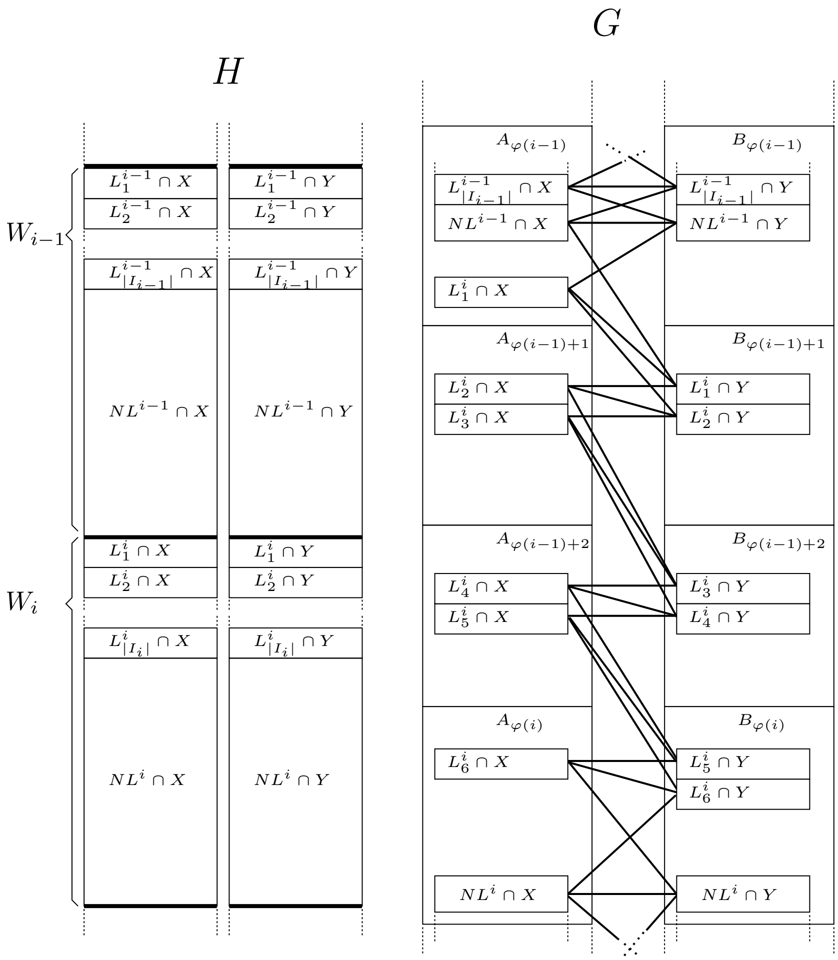

FIG. 5.1. *The linking procedure. On the right-hand side,* $|I_i| = 6$.

We now construct a map $f\colon V(H) \to V(C) = \{A_1, \dots, A_k, B_1, \dots, B_k\}$ by defining, for every $i \in [\ell]$,

$$f(x) := \begin{cases} A_{\varphi(i-1)+\lfloor j/2 \rfloor} & \text{if } x \in L_j^i \text{ with } j \in I_i, \\ A_{\varphi(i)} & \text{if } x \in NL^i, \end{cases} \tag{5.4}$$

for every $x \in W_i \cap X$, and

$$f(y) := \begin{cases} B_{\varphi(i-1)+\lceil j/2 \rceil} & \text{if } y \in L_j^i \text{ with } j \in I_i, \\ B_{\varphi(i)} & \text{if } y \in NL^i, \end{cases} \tag{5.5}$$

for every $y \in W_i \cap Y$, where all indices of $A's$ and $B's$ are to be taken modulo k. Directly from the construction, every $v \in V(H)$ is in exactly one W_i and then either in a L_j^i for exactly one $j \in I_i$ or in NL^i, so this is well-defined map on all of $V(H)$.

We now show that f is a graph homomorphism $H \to C$. To do this, we let an arbitrary edge $e \in F = E(H)$ be given and prove that the 2-set of the images of the two vertices in e under f is an edge of C. We will identify a set containing a single vertex with the vertex itself, so that we can write, e.g., $e \cap X \in I^i_j$.

To begin with, note that a set $\{A_i, B_{i'}\}$ is an edge of C if and only if the number $(i'-i) \mod k$, henceforward referred to as 'the difference', is 0 or 1. Moreover, it follows directly from the construction of L^i_j and NL^i (remember that we made sure that $|NL^i| \geq \beta n$) that exactly one of the following five statements is true.

(1) For exactly one $i \in [\ell]$ and exactly one $j \in I_i$, both vertices in e are in L^i_j.
(2) For exactly one $i \in [\ell]$ and exactly one $j \in I_i \backslash \{|I_i|\}$, one vertex in e is in L^i_j and one is in L^i_{j+1}.
(3) For exactly one $i \in [\ell]$, one vertex in e is in $L^i_{|I_i|}$ and one is in NL^i.
(4) For exactly one $i \in [\ell]$, both vertices in e are in NL^i.
(5) For exactly one $i \in [\ell]$, one vertex in e is in NL^i and one is in L^{i+1}_1.

If (1) is true, then $\{f(e \cap X), f(e \cap Y)\} = \{A_{\varphi(i-1)+\lfloor j/2 \rfloor}, B_{\varphi(i-1)+\lceil j/2 \rceil}\}$, and the difference is $(\lceil j/2 \rceil - \lfloor j/2 \rfloor) \mod k$, which is either 0 or 1 according to whether j is even or odd.

If (2) is true, then $\{f(e\cap X), f(e\cap Y)\}$ depends on whether it is $e\cap X$ or $e\cap Y$ that is in L^i_{j+1} but on nothing else. If $e\cap Y \in L^i_{j+1}$, then $e\cap X \in L^i_j$, hence $\{f(e\cap X), f(e\cap Y)\} = \{A_{\varphi(i-1)+\lfloor j/2 \rfloor}, B_{\varphi(i-1)+\lceil (j+1)/2 \rceil}\}$ and the difference is $(\lceil (j+1)/2 \rceil - \lfloor j/2 \rfloor) \mod k$, which always 1, whether j is even or odd. If $e \cap Y \in L^i_j$, then $e \cap X \in L^i_{j+1}$, hence $\{f(e \cap X), f(e \cap Y)\} = \{A_{\varphi(i-1)+\lfloor (j+1)/2 \rfloor}, B_{\varphi(i-1)+\lceil j/2 \rceil}\}$ and the difference is $(\lceil j/2 \rceil - \lfloor (j+1)/2 \rfloor) \mod k$, which is always 0, whether j is even or odd.

If (3) is true, then $\{f(e \cap X), f(e \cap Y)\}$ depends on whether it is $e \cap X$ or $e \cap Y$ that is in NL^i but on nothing else. If $e \cap Y \in NL^i$, then $e \cap X \in L^i_{|I_i|}$, hence $\{f(e \cap X), f(e \cap Y)\} = \{A_{\varphi(i-1)+\lfloor \frac{1}{2}|I_i| \rfloor}, B_{\varphi(i)}\} = \{A_{\varphi(i-1)+(\varphi(i)-\varphi(i-1)) \mod k}, B_{\varphi(i)}\} = \{A_{\varphi(i-1) \mod k+(\varphi(i)-\varphi(i-1)) \mod k}, B_{\varphi(i)}\} = \{A_{\varphi(i) \mod k}, B_{\varphi(i)}\} = \{A_{\varphi(i)}, B_{\varphi(i)}\}$, since all indices have been defined to be modulo k, hence the difference is 0. If $e\cap Y \in L^i_{|I_i|}$, then $e\cap X \in NL^i$, hence $\{f(e\cap X), f(e\cap Y)\} = \{A_{\varphi(i)}, B_{\varphi(i-1)+\lceil \frac{1}{2}|I_i| \rceil}\} = \{A_{\varphi(i)}, B_{\varphi(i-1)+(\varphi(i)-\varphi(i-1)) \mod k}\} = \{A_{\varphi(i)}, B_{\varphi(i)}\}$, analogously to the preceding calculation, hence the difference is 0 once more.

If (4) is true, then $\{f(e \cap X), f(e \cap Y)\} = \{A_{\varphi(i)}, B_{\varphi(i)}\}$, and the difference is 0.

If (5) is true, then $\{f(e\cap X), f(e\cap Y)\}$ depends on whether it is $e\cap X$ or $e\cap Y$ that is in L^{i+1}_1 but on nothing else. If $e\cap Y \in L^{i+1}_1$, then $e\cap X \in NL^i$, hence $\{f(e\cap X), f(e\cap Y)\} = \{A_{\varphi(i)}, B_{\varphi((i+1)-1)+\lceil 1/2 \rceil}\} = \{A_{\varphi(i)}, B_{\varphi(i)+1}\}$, hence the difference is 1. If $e \cap X \in L^{i+1}_1$, then $e\cap Y \in NL^i$, hence $\{f(e\cap X), f(e\cap Y)\} = \{A_{\varphi((i+1)-1)+\lfloor 1/2 \rfloor}, B_{\varphi(i)}\} = \{A_{\varphi(i)}, B_{\varphi(i)}\}$, hence the difference is 0.

Since in all possible cases, the difference is 0 or 1, we have completed the proof that f is a graph homomorphism $H \to C$.

We now prove (H1) and (H2). Define $S := \bigcup_{i\in[\ell]} L^i$. Then $|S| \leq \ell \cdot 2k \cdot \beta n \leq \ell \cdot 2k \cdot (\xi'/(2\ell k)) \cdot n = \xi' n \leq \xi n$, which shows (H1), and (H2) is obvious from the definitions of S and the map f above.

We now prove (H3). For this it suffices to note, rather crudely, that for every $j \in [k]$, no pre-image $f^{-1}(A_j)$ can become larger than the sum of the sizes of all sets W_i assigned to A_j by φ (which by the definition of f equals the sum of all $x_i = |X \cap W_i|$ with $\varphi(i) = j$) plus the total number of linking vertices, i.e. for every $j \in [k]$, using the choice of β and using that φ has the property promised by Lemma 5.1, we have $|f^{-1}(A_j)| \leq \left(\sum_{i\in\varphi^{-1}(j)} x_i\right) + |\bigcup_{i\in[\ell]} L^i| \leq n_j + \xi' n + \ell \cdot |L^i| = n_j + \xi' n + 2k\ell\beta n \leq n_j + 2\xi' n = n_j + \xi n$, completing the proof of (H3). □

6. Concluding remarks. Unbalanced H and G. Essentially the same proof allows for an analogue of Theorem 1.3 for bipartite graphs H and G that are not balanced but whose colour classes have the same sizes. More precisely, let $H = (X\dot{\cup}Y, F)$ and $G = (A\dot{\cup}B, E)$ be as in Theorem 1.3, except that $|X| = |A| = n_1$ and $|Y| = |B| = n_2$ (where $n_1 + n_2 = 2n$) and the minimum degree condition on G is replaced by the following condition. For all $v \in A$ we have $\deg_G(v) \geq (\frac{1}{2} + \gamma)n_2$ and for all $w \in B$ we have $\deg_G(w) \geq (\frac{1}{2} + \gamma)n_1$. Then H is a subgraph of G.

Thresholds for r-partite H and G. We believe that the following r-partite analogues of our main result might be true and susceptible to similar methods as those used in this paper.

(1) For all r, γ and Δ there is a positive constant β and an integer n_0 such that for all $n \geq n_0$ the following holds. Let G and H *both* be *balanced* r-partite graphs on n vertices such that G has minimum degree $\delta(G) \geq (\frac{2r-3}{2r} + \gamma)n$ and H has maximum degree at most Δ and bandwidth at most βn. Then G contains a copy of H.

(2) Same formulation as (1), but now G and H are allowed to be *arbitrary* r-partite graphs having *compatible sizes of partition classes* (an obvious necessary condition) while the minimum degree threshold is raised to $\delta(G) \geq (\frac{3r-5}{3r-2} + \gamma)n$.

Concerning (1), interested readers are encouraged to compare the relevant articles of Magyar and Martin [38], and Martin and Szemerédi [39] who considered sufficient degree conditions for the existence of K^r-factors in balanced r-partite graph for $r = 3$ and $r = 4$.

Both statements, if true, would be essentially best-possible in the sense that replacing γ by 0 makes them false. This is witnessed by the following example: As to (1), start with a *balanced* complete r-partite graph with k vertices in each class, delete all the edges of exactly one of the complete bipartite graphs in it and replace them by the edges of two vertex-disjoint complete balanced bipartite graphs, one having $\lfloor k/2 \rfloor$, the other $\lceil k/2 \rceil$ vertices on either side. It is not difficult to see that this graph G does not contain an $(r-1)$-th power of a Hamilton cycle. As to (2), modify the example just described by starting with a non-balanced complete r-partite graph G having cluster sizes of $(r-2)$-times $\frac{3}{3r-2}n$ and 2-times $\frac{2}{3r-2}n$ and take the two smaller classes as the ones supporting the special bipartite graph. Moreover, by taking a certain H similar to the $(r-1)$-th power of a Hamilton cycle, it is not difficult to define an admissible H meeting the requirement of compatible sizes of partition classes compared to G (an $(r-1)$-th power of a Hamilton cycle does not, and is therefore no longer a valid example) which is nevertheless not contained in G for similar reasons as before.

Generating systems for the cycle space. As an application of Theorem 1.3 one can show the following result. For every $\gamma > 0$ there is $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ every balanced bipartite graph G on $2n$ vertices with $\delta(G) \geq (\frac{1}{2} + \gamma)n$ has the property that the edge-sets of all Hamilton cycles in G form a generating system for the cycle space of G. A proof for this will be given in a forthcoming paper [20]. The proof strategy is to first show that a specific spanning subgraph H of bounded maximum degree and bandwidth has this property and then show (using a result of Locke [37]) that the property transfers to the whole graph G.

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