Induced C_5-free graphs of fixed density: counting and homogeneous sets

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Abstract

For $c \in (0,1)$ let $\mathcal{P}_n(c)$ denote the set of n-vertex perfect graphs with density c and $\mathcal{C}_n(c)$ the set of n-vertex graphs without induced C_5 and with density c. We show that $\log_2 |\mathcal{P}_n(c)|/\binom{n}{2} = \log_2 |\mathcal{C}_n(c)|/\binom{n}{2} = h(c) + o(1)$ with $h(c) = \frac{1}{2}$ if $\frac{1}{4} \leq c \leq \frac{3}{4}$ and $h(c) = \frac{1}{2}H(|2c-1|)$ otherwise, where H is the binary entropy function.

Furthermore, we use this result to deduce that almost all graphs in $\mathcal{C}_n(c)$ have homogeneous sets of linear size. This answers a special case of a question raised by Loebl et al.

Keywords: graph theory, extremal problems, asymptotic counting, homogeneous sets

1 Introduction and results

In this paper we investigate classes of graphs that are defined by forbidding certain substructures. Let $\mathcal{H}$ be such a class. We focus on two related goals: to approximate the cardinality of $\mathcal{H}$ and to determine the structure of a typical graph in $\mathcal{H}$. In particular, we add the additional constraint that all graphs in $\mathcal{H}$ must have the same density c and would like to know how the answer to these questions depends on the parameter c.

We consider induced C_5-free graphs of density c and provide bounds for their number. In [12] Prömel and Steger proved that the class of induced C_5-free graphs is closely related to two other classes of graphs: perfect graphs on the one hand and generalised split graphs on the other. In the spirit of their results we relate the sizes of these three classes when restricted to graphs of given density c.

With $\mathcal{G}_n(c)$ being the set of all graphs on vertex set $[n]$ with $c\binom{n}{2}$ edges and $\mathcal{F}orb^*_n(F)$ being the set of all graphs on vertex set $[n]$ that do not contain an induced copy of F, we define the following graph classes:

$$\begin{aligned}
\mathcal{C}(n,c) &:= \mathcal{G}_n(c) \cap \mathcal{F}orb^*_n(C_5)\,, \\
\mathcal{P}(n,c) &:= \big\{G \in \mathcal{G}_n(c) \colon G \text{ is perfect}\big\}\,, \\
\mathcal{S}(n,c) &:= \big\{G \in \mathcal{G}_n(c) \colon G \text{ is a generalised split graph}\big\}\,,
\end{aligned}$$

where $G = (V, E)$ is defined to be a generalised split graph if G or its complement admits a partition $V = V_1 \dot\cup \ldots \dot\cup V_k$ such that $G[V_i]$ is a clique and for $i > j > 1$ we have $e(V_i, V_j) = 0$.

Observe that for all n and $c \in (0, 1)$ we have $\mathcal{S}(n,c) \subset \mathcal{P}(n,c) \subset \mathcal{C}(n,c)$. Our first result states that the cardinalities of these three sets are equal up to a multiplicative term of order $2^{o(n^2)}$ and can be described by the following function (see also Figure 1). Let

$$h(c) := \begin{cases} H(2c)/2 & \text{if } 0 < c < \frac{1}{4}\,, \\ 1/2 & \text{if } \frac{1}{4} \le c \le \frac{3}{4}\,, \\ H(2c-1)/2 & \text{if } \frac{3}{4} < c < 1\,, \end{cases}$$

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where $H(x)$ is the *binary entropy* function, that is, for $x \in (0,1)$ we set $H(x) := -x\log_2 x - (1-x)\log_2(1-x)$.

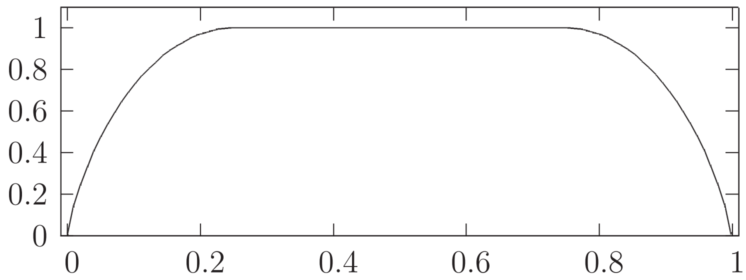

Fig. 1. Values of $h(c)$ for $c \in (0,1)$.

Theorem 1.1 *For all $c \in (0,1)$ we have*

$$\lim_{n\to\infty} \frac{\log_2 |\mathcal{C}(n,c)|}{\binom{n}{2}} = \lim_{n\to\infty} \frac{\log_2 |\mathcal{P}(n,c)|}{\binom{n}{2}} = \lim_{n\to\infty} \frac{\log_2 |\mathcal{S}(n,c)|}{\binom{n}{2}} = h(c)\,.$$

Let us now move from the question of approximating cardinalities to determining the structure of a typical element in $\mathcal{F}orb^*_n(C_5)$. A well-known conjecture by Erdős and Hajnal [7] states that any family of graphs that does not contain a certain fixed graph F as an induced subgraph must contain a homogeneous set, i.e., a clique or a stable set, whose size is polynomial in the number of vertices. The conjecture is known to be true only for few graphs F, but open already for $F = C_5$ (see [9]). However, Loebl, Reed, Scott, Thomason, and Thomassé [10] recently showed that for any graph F *almost all graphs* in $\mathcal{F}orb^*_n(F)$ have a polynomially sized homogeneous set. Moreover, they asked for which graphs F it is true that almost all graphs in $\mathcal{F}orb^*_n(F)$ do have a *linearly* sized homogeneous set.

It may seem at first sight that our estimates derived in Theorem 1.1 are too rough to tell us something about the structure of almost all graphs in $\mathcal{F}orb^*_n(C_5)$. However, we can combine them with the ideas of [10] to answer the question of Loebl et al. in the affirmative for the case $F = C_5$. In fact, we can prove this assertion even in the case where we, again, restrict the class to graphs with a given density, that is, to $\mathcal{F}orb^*_n(C_5,c) = \mathcal{G}_n(c) \cap \mathcal{F}orb^*_n(C_5)$.

Theorem 1.2 *For $\eta > 0$ denote by $\mathcal{F}orb^*_{n,\eta}(F,c)$ the set of graphs $G \in \mathcal{F}orb^*_n(F,c)$ without a homogeneous set of size at least ηn. Then for every $0 < c < 1$ there exists $\eta > 0$ such that*

$$\lim_{n\to\infty} \frac{|\mathcal{F}orb^*_{n,\eta}(C_5,c)|}{|\mathcal{F}orb^*_n(C_5,c)|} = 0\,.$$

2 Background

In this chapter we give a brief overview of how our results relate to earlier counting results for classes $\mathcal{H}$ defined by forbidden (induced) structures. The quantity $|\mathcal{H}_n|$ where $\mathcal{H}_n := \{G \in \mathcal{H} : V(G) = [n]\}$ is called the *speed* of $\mathcal{H}$. Often exact formulas or good estimates for $|\mathcal{H}_n|$ are out of reach. In these cases, however, one might still ask for the asymptotic behaviour of the speed of $\mathcal{H}$. One prominent result in this direction was obtained by Erdős, Frankl and Rödl [6] who considered properties $\mathcal{F}orb(F)$ defined by a single forbidden (weak) subgraph F. They proved that for each graph F with $\chi(F) \geq 3$ the class $\mathcal{F}orb_n(F)$ of n-vertex graphs that do not contain F as a subgraph satisfies $|\mathcal{F}orb_n(F)| = 2^{\mathrm{ex}(F,n)+o(n^2)}$ where $\mathrm{ex}(F,n) := (\chi(F)-2)\binom{n}{2}/(\chi(F)-1)$. In other words, if $\chi(F) \geq 3$ then the speed of $\mathcal{F}orb(F)$ asymptotically only depends on the chromatic number of F.

As explained in Section 1, we are interested in features of the picture at a more fine grained scale. Let $\mathcal{F}orb_n(F,c) = \mathcal{F}orb_n(F) \cap \mathcal{G}_n(c)$. Straightforward modifications of the proof of the theorem of Erdős, Frankl and Rödl [6] yield the following bounds for $|\mathcal{F}orb_n(F,c)|$ with F being a graph with $\chi(F) = r$ and $c \in (0, \frac{r-2}{r-1})$. We have

$$\lim_{n\to\infty} \frac{\log_2 |\mathcal{F}orb_n(F,c)|}{\binom{n}{2}} = \tfrac{r-2}{r-1}\, H\!\left(\tfrac{r-1}{r-2}c\right).$$

Notice that $\lim_{n\to\infty} \left(\log_2 |\mathcal{F}orb_n(F,c)|\right)/\binom{n}{2} = 0$ for $c \geq \frac{r-2}{r-1}$ by the theorem of Erdős and Stone [8].

Determining $|\mathcal{F}orb^*_n(F)|$ is more challenging and was first considered by Prömel and Steger [13]. They specified a graph parameter, the so-called colouring number $\chi^*(F)$ of F, that serves as a suitable replacement of the chromatic number in the theorem of Erdős, Frankl and Rödl. More precisely, they showed that $|\mathcal{F}orb^*_n(F)| = 2^{\mathrm{ex}^*(F,n)+o(n^2)}$ with $\mathrm{ex}^*(F,n) := \big(\chi^*(F)-2\big)\binom{n}{2}/\big(\chi^*(F)-1\big)$ where $\chi^*(F)$ is defined as follows. A *generalised r-colouring* of F with $r' \in [0,r]$ cliques is a partition of $V(F)$ into r' cliques and $r-r'$ independent sets. The *colouring number* $\chi^*(F)$ is the largest integer $r+1$ such that there is an $r' \in [r]$ for which F has no generalised r-colouring with r' cliques. Alekseev [1], and Bollobás and Thomason [4] generalised the result of Prömel and Steger to arbitrary hereditary graph properties $\mathcal{H}$ (i.e., graph classes which are closed under isomorphism and taking induced subgraphs). More precise estimates for the speed of $\mathcal{H}$ were given by Alon, Balogh, Bollobás, and Morris [2]. For the case of $\mathcal{F}orb^*_n(C_4)$ and $\mathcal{F}orb^*_n(C_5)$ the speed can in fact be approximated up to a factor of $2^{O(n)}$ and it can be shown that almost all graphs in $\mathcal{F}orb^*_n(C_5)$ are generalised split graphs [11,12].

Bollobás and Thomason [5] studied the probability $\mathbb{P}_{\mathcal{H}} := \mathbb{P}[\mathcal{G}(n,p) \in \mathcal{H}]$ of an arbitrary hereditary property $\mathcal{H}$ in the probability space $\mathcal{G}(n,p)$. In this context, our Theorem 1.1 estimates the probability of $\mathcal{H} = \mathcal{F}orb^*_n(C_5)$ in the probability space $\mathcal{G}(n,m)$ with $m = c\binom{n}{2}$.

3 Sketch of proofs

The proofs of Theorem 1.1 and Theorem 1.2 use the regularity method for induced subgraphs and ideas from [5] and [10]. Here we summarise the main steps of the proof of Theorem 1.2.

For each graph G in $\mathcal{F}orb^*_{n,\eta}(C_5, c)$ we apply the regularity lemma to obtain a reduced graph $R = R(G)$ of bounded size. We encode densities of regular pairs by colouring the edges in R white, grey or black depending on whether the regular pairs have density in $[0, \delta]$, $(\delta, 1-\delta)$ or $[1-\delta, 1]$. The resulting coloured graph is called a *type*. The crucial observation is that R cannot have a grey triangle as this would force an induced copy of C_5 in G (see, e.g., the embedding lemma for induced subgraphs in [3]). For a fixed type R we then obtain an upper bound for the number of graphs $G \in \mathcal{F}orb^*_{n,\eta}(C_5, c)$ with type R which only depends on (the density c and) the number of grey edges in R through the following observations.

We consider each cluster V_i of a partition of $V(G)$ corresponding to R separately. We show that the fact that G does not contain homogeneous sets of size ηn implies that $G[V_i]$ has a linear number of vertex-disjoint induced copies of P_3, the path on three vertices, or a linear number of vertex-disjoint induced copies of the anti-path $\overline{P}_3$, the complement of P_3. We next prove that many induced copies of P_3 or $\overline{P}_3$ in two clusters V_i and V_j, however, limit the number of possibilities to insert edges between V_i and V_j without inducing a C_5.

Using the fact that the number of grey edges in R is bounded by Turán's theorem we thus obtain an upper bound on $|\mathcal{F}orb^*_{n,\eta}(C_5, c)|$. Together with the estimate for $|\mathcal{F}orb^*_n(C_5, c)|$ from Theorem 1.1 this yields the desired result.

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