Bandwidth, expansion, treewidth, separators, and universality for bounded degree graphs☆

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Abstract

We establish relations between the bandwidth and the treewidth of bounded degree graphs G, and relate these parameters to the size of a separator of G as well as the size of an expanding subgraph of G. Our results imply that if one of these parameters is sublinear in the number of vertices of G then so are all the others. This implies for example that graphs of fixed genus have sublinear bandwidth or, more generally, a corresponding result for graphs with any fixed forbidden minor. As a consequence we establish a simple criterion for universality for such classes of graphs and show for example that for each $\gamma > 0$ every n-vertex graph with minimum degree $(\frac{3}{4} + \gamma)n$ contains a copy of every bounded-degree planar graph on n vertices if n is sufficiently large.

1. Introduction and Results

There are a number of different parameters in graph theory which measure how well a graph can be organized in a particular way, where the type of desired arrangement is often motivated by geometrical properties, algorithmic

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considerations or specific applications. Well-known examples of such parameters are the genus, the bandwidth, or the treewidth of a graph. While the the genus characterizes the surfaces on which a particular graph can be drawn without crossings, the other two parameters describe how well a graph can be laid out in a path-like, respectively tree-like, manner. The central topic of this paper is to discuss the relations between such parameters. We would like to determine how they influence each other and what causes them to be large. To this end we will mostly be interested in distinguishing between the case where these parameters are linear in n, where n is the number of vertices in the graph under investigation, and the case where they are sublinear in n.

Let $G = (V, E)$ be a graph on n vertices. The *bandwidth* of G is denoted by $\mathrm{bw}(G)$ and defined to be the minimum positive integer b, such that there exists a labelling of the vertices in V by numbers $1, \ldots, n$ so that the labels of every pair of adjacent vertices differ by at most b. Clearly one reason for a graph to have high bandwidth are vertices of high degree. Thus the star $K_{1,n-1}$ illustrates that in general even trees may have a bandwidth of order $\Omega(n)$. In [8] Chung proved however that any n-vertex tree T with maximum degree Δ has bandwidth at most $5n/\log_\Delta(n)$. Our first theorem extends Chung's result to planar graphs.

Theorem 1. *Suppose $\Delta \geq 4$. Let G be a planar graph on n vertices with maximum degree at most Δ. Then the bandwidth of G is bounded from above by*

$$\mathrm{bw}(G) \leq \frac{15n}{\log_\Delta(n)}.$$

It is easy to see that the bound in Theorem 1 is sharp up to the multiplicative constant—since the bandwidth of any graph G is bounded from below by $(n-1)/\mathrm{diam}(G)$, it suffices to consider for example the complete binary tree on n vertices. (We remark in passing that Theorem 1 implies that graphs with maximum degree $1 \leq \Delta \leq 3$ must satisfy an upper bound of $\mathrm{bw}(G) \leq 20n/\log_\Delta(n)$.) Theorem 1 is used in [7] to infer a result about the geometric realizability of planar graphs $G = (V, E)$ with $|V| = n$ and $\Delta(G) \leq \Delta$. In Section 2.2 we will consider further applications of Theorem 1 and similar results to the domain of universal graphs and derive implications such as the following. If n is sufficiently large then any n-vertex graph with minimum degree slightly above $\frac{3}{4}n$ contains every planar n-vertex graphs with bounded maximum degree (cf. Corollary 15) as a subgraph.

Theorem 1 states that planar graphs of bounded maximum degree have sublinear bandwidth. Similar results can be formulated for graphs of any fixed genus and, more generally, for any graph class defined by a fixed set of forbidden minors (see Section 2.1). It is also clear that not all graphs of bounded maximum degree have sublinear bandwidth: Consider for example a random bipartite graph G on n vertices with bounded maximum degree. Indeed, with high probability, G cannot have small bandwidth since in any linear ordering of its vertices there will be an edge between the first $n/4$ and the last $n/4$ vertices in this ordering. The reason for this obstacle is that G has good expansion properties (definitions and exact statements are provided below). This implies that graphs with sublinear bandwidth (as in Theorem 1) cannot exhibit good expansion properties. One may ask whether the converse is also true, i.e. whether the absence of big expanding subgraphs in bounded-degree graphs must lead to small bandwidth. We will prove that this is indeed the case via the existence of certain separators.

In fact, we will show a more general theorem (Theorem 9) which proves that the concepts of sublinear bandwidth, sublinear treewidth, bad expansion properties, and sublinear separators are equivalent for graphs of bounded maximum degree. In order to establish this theorem, we will now discuss quantitative relations between the parameters involved. Since planar graphs are known to have small separators [23], we will get Theorem 1 as a byproduct of these results in Section 2.1.

Let us introduce some more definitions to state our further results. For a graph $G = (V, E)$ and disjoint vertex sets $A, B \subseteq V$ we denote by $E(A, B)$ the set of edges with one vertex in A and one vertex in B and by $e(A, B)$ the number of such edges. Next, we will introduce the notions of *tree decomposition* and *treewidth*. Roughly speaking, a tree decomposition tries to arrange the vertices of a graph in a tree-like manner and the treewidth measures how well this can be done.

Definition 2 (treewidth). *A tree decomposition of a graph $G = (V, E)$ is a pair $(\{X_i : i \in I\},\ T = (I, F))$ where $\{X_i : i \in I\}$ is a family of subsets $X_i \subseteq V$ and $T = (I, F)$ is a tree such that the following holds:*

(a) $\bigcup_{i \in I} X_i = V$,
(b) *for every edge $\{v, w\} \in E$ there exists $i \in I$ with $\{v, w\} \subseteq X_i$,*
(c) *for every $i, j, k \in I$: if j lies on the path from i to k in T, then $X_i \cap X_k \subseteq X_j$.*

The width of $(\{X_i : i \in I\}, T = (I, F))$ *is defined as* $\max_{i \in I} |X_i| - 1$. *The treewidth* $\mathrm{tw}(G)$ *of* G *is the minimum width of a tree decomposition of* G.

It follows directly from the definition that $\mathrm{tw}(G) \leq \mathrm{bw}(G)$ for any graph G: if the vertices of G are labelled by numbers $1, \ldots, n$ such that the labels of adjacent vertices differ by at most b, then $I := [n - b]$, $X_i := \{i, \ldots, i + b\}$ for $i \in I$ and $T := (I, F)$ with $F := \{\{i - 1, i\} : 2 \leq i \leq n - b\}$ define a tree decomposition of G with width b.

A *separator* in a graph is a small cut-set that splits the graph into components of limited size.

Definition 3 (separator, separation number). *Let* $0 < \alpha < 1$ *be a real number,* $s \in \mathbb{N}$ *and* $G = (V, E)$ *a graph. A subset* $S \subseteq V$ *is said to be an* (s, α)*-separator of* G*, if there exist subsets* $A, B \subseteq V$ *such that*

(a) $V = A \,\dot\cup\, B \,\dot\cup\, S$,
(b) $|S| \leq s$, $|A|, |B| \leq \alpha |V|$, *and*
(c) $E(A, B) = \emptyset$.

We also say that S *separates* G *into* A *and* B. *The separation number* $\mathrm{s}(G)$ *of* G *is the smallest* s *such that all subgraphs* G' *of* G *have an* $(s, 2/3)$*-separator.*

A vertex set is said to be expanding, if it has many external neighbours. We call a graph *bounded*, if every sufficiently large subgraph contains a subset which is not expanding.

Definition 4 (expander, bounded). *Let* $\varepsilon > 0$ *be a real number,* $b \in \mathbb{N}$ *and consider graphs* $G = (V, E)$ *and* $G' = (V', E')$. *We say that* G' *is an* ε*-expander if all subsets* $U \subseteq V'$ *with* $|U| \leq |V'|/2$ *fulfil* $|N(U)| \geq \varepsilon |U|$. *(Here* $N(U)$ *is the set of neighbours of vertices in* U *that lie outside of* U.*) The graph* G *is called* (b, ε)*-bounded, if no subgraph* $G' \subseteq G$ *with* $|V'| \geq b$ *vertices is an* ε*-expander. Finally, we define the* ε*-boundedness* $\mathrm{b}_\varepsilon(G)$ *of* G *to be the minimum* b *for which* G *is* $(b + 1, \varepsilon)$*-bounded.*

There is a wealth of literature on this class of graphs (see e.g. [16]). In particular, it is known that for example (bipartite) random graphs with bounded maximum degree form a family of ε-expanders. We also loosely say that such graphs have *good expansion properties*. As indicated earlier, our aim is to provide relations between the parameters we defined above. A well known example of a result of this type is the following theorem due to Robertson and Seymour which relates the treewidth and the separation number of a graph.[2]

Theorem 5 (treewidth→separator, [24]). *All graphs G have separation number*

$$\mathrm{s}(G) \leq \mathrm{tw}(G) + 1.$$

This theorem states that graphs with small treewidth have small separators. By repeatedly extracting separators, one can show that (a qualitatively different version of) the converse also holds: $\mathrm{tw}(G) \leq \mathcal{O}(\mathrm{s}(G) \log n)$ for a graph G on n vertices (see e.g. [4], Theorem 20). In this paper, we use a similar but more involved argument to show that one can establish the following relation linking the separation number with the bandwidth of graphs with bounded maximum degree.

Theorem 6 (separator→bandwidth). *For each $\Delta \geq 4$ every graph G on n vertices with maximum degree $\Delta(G) \leq \Delta$ has bandwidth*

$$\mathrm{bw}(G) \leq \frac{6n}{\log_\Delta(n/\,\mathrm{s}(G))}.$$

The proof of this theorem is provided in Section 3.1. Observe that Theorems 5 and 6 together with the obvious inequality $\mathrm{tw}(G) \leq \mathrm{bw}(G)$ tie the concepts of treewidth, bandwidth, and separation number well together. Apart from the somewhat negative statement of *not having* a small separator, what can we say about a graph with large tree- or bandwith? The next theorem states that such a graph must contain a big expander.

Theorem 7 (bounded→treewidth). *Let $\varepsilon > 0$ be constant. All graphs G on n vertices have treewidth $\mathrm{tw}(G) \leq 2\,\mathrm{b}_\varepsilon(G) + 2\varepsilon n$.*

A result with similar implications was recently proved by Grohe and Marx in [14]. It shows that $b_\varepsilon(G) < \varepsilon n$ implies $\mathrm{tw}(G) \leq 2\varepsilon n$. For the sake of being self contained we present our (short) proof of Theorem 7 in Section 3.2. In addition, it is not difficult to see that conversely the boundedness of a graph can be estimated via its bandwidth—which we prove in Section 3.2, too.

[2] In fact, their result states that any graph G has a $(\mathrm{tw}(G) + 1, 1/2)$-separator, and doesn't talk about subgraphs of G. But since every subgraph of G has treewidth at most $\mathrm{tw}(G)$, it thus also has a $(\mathrm{tw}(G)+1, 1/2)$-separator and the result, as stated here, follows.

Proposition 8 (bandwidth→bounded). *Let $\varepsilon > 0$ be constant. All graphs G on n vertices have* $\mathrm{b}_\varepsilon(G) \leq 2\,\mathrm{bw}(G)/\varepsilon$.

A qualitative consequence summarizing the four results above is given in the following theorem. It states that if one of the parameters bandwidth, treewidth, separation number, or boundedness is sublinear for a family of graphs, then so are the others.

Theorem 9 (sublinear equivalence theorem). *Let Δ be an arbitrary but fixed positive integer and consider a hereditary class of graphs $\mathcal{C}$ such that all graphs in $\mathcal{C}$ have maximum degree at most Δ. Denote by $\mathcal{C}_n$ the set of those graphs in $\mathcal{C}$ with n vertices. Then the following four properties are equivalent:*

(1) *For all $\beta_1 > 0$ there is n_1 s.t.* $\mathrm{tw}(G) \leq \beta_1 n$ *for all $G \in \mathcal{C}_n$ with $n \geq n_1$.*
(2) *For all $\beta_2 > 0$ there is n_2 s.t.* $\mathrm{bw}(G) \leq \beta_2 n$ *for all $G \in \mathcal{C}_n$ with $n \geq n_2$.*
(3) *For all $\beta_3, \varepsilon > 0$ there is n_3 s.t.* $\mathrm{b}_\varepsilon(G) \leq \beta_3 n$ *for all $G \in \mathcal{C}_n$ with $n \geq n_3$.*
(4) *For all $\beta_4 > 0$ there is n_4 s.t.* $\mathrm{s}(G) \leq \beta_4 n$ *for all $G \in \mathcal{C}_n$ with $n \geq n_4$.*

Proof. (1)⇒(4): Given $\beta_4 > 0$ set $\beta_1 := \beta_4/2$, let n_1 be the constant from (1) for this β_1, and set $n_4 := \max\{n_1, 2/\beta_4\}$. Now consider $G \in \mathcal{C}_n$ with $n \geq n_4$. By assumption we have $\mathrm{tw}(G) \leq \beta_1 n$ and thus we can apply Theorem 5 to conclude that $\mathrm{s}(G) \leq \mathrm{tw}(G) + 1 \leq \beta_1 n + 1 \leq (\beta_4/2 + 1/n)n \leq \beta_4 n$.

(4)⇒(2): Given $\beta_2 > 0$ let $d := \min\{4, \Delta\}$, set $\beta_4 := d^{-6/\beta_2}$, get n_4 from (4) for this β_4, and set $n_2 := n_4$. Let $G \in \mathcal{C}_n$ with $n \geq n_2$. We conclude from (4) and Theorem 5 that

$$\mathrm{bw}(G) \leq \frac{6n}{\log_d n - \log_d \mathrm{s}(G)} \leq \frac{6n}{\log_d n - \log_d(d^{-6/\beta_2} n)} = \beta_2 n.$$

(2)⇒(3): Given $\beta_3, \varepsilon > 0$ set $\beta_2 := \varepsilon\beta_3$, get n_2 from (2) for this β_2 and set $n_3 := n_2$. By (2) and Proposition 8 we get for $G \in \mathcal{C}_n$ with $n \geq n_3$ that $\mathrm{b}_\varepsilon(G) \leq 2\,\mathrm{bw}(G)/\varepsilon \leq 2\beta_2 n/\varepsilon \leq \beta_3 n$.

(3)⇒(1): Given $\beta_1 > 0$, set $\beta_3 := \beta_1/4$, $\varepsilon := \beta_1/4$ and get n_3 from (3) for this β_3 and ε, and set $n_1 := n_3$. Let $G \in \mathcal{C}_n$ with $n \geq n_4$. Then (3) and Theorem 7 imply $\mathrm{tw}(G) \leq 2\,\mathrm{b}_\varepsilon(G) + 2\varepsilon n \leq 2\beta_3 n + 2(\beta_1/4)n = \beta_1 n$.

2. Applications

For many interesting bounded degree graph classes (non-trivial) upper bounds on the bandwidth are not at hand. A wealth of results however has been obtained about the existence of sublinear separators. This illustrates the importance of Theorem 9. In this section we will give examples of such separator theorems and provide applications of them in conjunction with Theorem 9.

2.1. Separator theorems

A classical result in the theory of planar graphs concerns the existence of separators of size $2\sqrt{2n}$ in any planar graph on n vertices proven by Lipton and Tarjan [23] in 1977. Clearly, together with Theorem 6 this result implies Theorem 1 from the introduction. This motivates why we want to consider some generalizations of the planar separator theorem in this section. The first such result is due to Gilbert, Hutchinson, and Tarjan [13] and deals with graphs of arbitrary genus. [3]

Theorem 10 ([13]). *An n-vertex graph G with genus $g \geq 0$ has separation number $\mathrm{s}(G) \leq 6\sqrt{gn} + 2\sqrt{2n}$.*

For fixed g the class of all graphs with genus at most g is closed under taking minors. Here H is a minor of G if it can be obtained from G by a sequence of edge deletions and contractions. A graph G is called H-minor free if H is no minor of G. The famous graph minor theorem by Robertson and Seymour [25] states that any minor closed class of graphs can be characterized by a finite set of forbidden minors (such as $K_{3,3}$ and K_5 in the case of planar graphs). The next separator theorem by Alon, Seymour, and Thomas [2] shows that already forbidding one minor enforces a small separator.

Theorem 11 ([2]). *Let H be an arbitrary graph. Then any n-vertex graph G that is H-minor free has separation number $\mathrm{s}(G) \leq |H|^{3/2}\sqrt{n}$.*

We can apply these theorems to draw the following conclusion concerning the bandwidth of bounded-degree graphs with fixed genus or some fixed forbidden minor from Theorem 6.

Corollary 12. *Let g be a positive integer, $\Delta \geq 4$ and H be an h-vertex graph and G an n-vertex graph with maximum degree $\Delta(G) \leq \Delta$.*

[3] Again, the separator theorems we refer to bound the size of a separator in G. Since the class of graphs with genus less than g (or, respectively, of H-minor free graphs) is closed under taking subgraphs however, this theorem can also be applied to such subgraphs and thus the bound on $\mathrm{s}(G)$ follows.

(a) *If G has genus g then* $\mathrm{bw}(G) \leq 15n/\log_\Delta(n/g)$.
(b) *If G is H-minor free then* $\mathrm{bw}(G) \leq 12n/\log_\Delta(n/h^3)$.

2.2. Embedding problems and universality

A graph H that contains copies of all graphs $G \in \mathcal{G}$ for some class of graphs $\mathcal{G}$ is also called *universal for* $\mathcal{G}$. The construction of sparse universal graphs for certain families $\mathcal{G}$ has applications in VLSI circuit design and was extensively studied (see e.g. [1] and the references therein). In contrast to these results our focus is not on minimizing the number of edges of H, but instead we are interested in giving a relatively *simple* criterion for universality for $\mathcal{G}$ that is satisfied by *many* graphs H of the same order as the largest graph in $\mathcal{G}$.

The setting with which we are concerned here are embedding results that guarantee that a bounded-degree graph G can be embedded into a graph H with sufficiently high minimum degree, even when G and H have the same number of vertices. Dirac's theorem [10] concerning the existence of Hamiltonian cycles in graphs of minimum degree $n/2$ is a classical example for theorems of this type. It was followed by results of Corrádi and Hajnal [9], Hajnal and Szemerédi [15] about embedding K_r-factors, and more recently by a series of theorems due to Komlós, Sarközy, and Szemerédi and others [12; 18; 19; 20; 21; 26] which deal with powers of Hamiltonian cycles, trees, and H-factors. Along the lines of these results the following unifying conjecture was made by Bollobás and Komlós [17] and recently proven by Böttcher, Schacht, and Taraz [5].

Theorem 13 ([5]). *For all $r, \Delta \in \mathbb{N}$ and $\gamma > 0$, there exist constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the following holds. If G is an r-chromatic graph on n vertices with $\Delta(G) \leq \Delta$ and bandwidth at most βn and if H is a graph on n vertices with minimum degree $\delta(H) \geq (\frac{r-1}{r} + \gamma)n$, then G can be embedded into H.*

The proof of Theorem 13 heavily uses the bandwidth constraint insofar as it constructs the required embedding sequentially following the ordering given by the vertex labels of G. Here it is of course beneficial that the neighbourhood of every vertex v in G is confined to the βn vertices which preceed or immediately follow v.

Also, it is not difficult to see that the statement in Theorem 13 becomes false without the constraint on the bandwidth : Consider $r = 2$, let G be a random bipartite graph with bounded maximum degree and let H be the graph formed by two cliques of size $(1/2 + \gamma)n$ each, which share exactly $2\gamma n$ vertices. Then H cannot contain a copy of G, since in G every vertex set of size $(1/2 - \gamma)n$ has more than $2\gamma n$ neighbours. The reason for this obstruction is again that G has good expansion properties.

On the other hand, Theorem 9 states that in bounded degree graphs, the existence of a big expanding subgraph is in fact the only obstacle which can prevent sublinear bandwidth and thus the only possible obstruction for a universality result as in Theorem 13. More precisely we immediately get the following corollary from Theorem 9.

Corollary 14. *If the class $\mathcal{C}$ meets one (and thus all) of the conditions in Theorem 9, then the following is also true. For every $\gamma > 0$ and $r \in \mathbb{N}$ there exists n_0 such that for all $n \geq n_0$ and for every graph $G \in \mathcal{C}_n$ with chromatic number r and for every graph H on n vertices with minimum degree at least $(\frac{r-1}{r} + \gamma)n$, the graph H contains a copy of G.*

By Theorem 1 we infer as a special case that all sufficiently large graphs with minimum degree $(\frac{3}{4} + \gamma)n$ are universal for the class of bounded-degree planar graphs. Universal graphs for bounded degree planar graphs have also been studied in [3; 6].

Corollary 15. *For all $\Delta \in \mathbb{N}$ and $\gamma > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the following holds:*

(a) *Every 3-chromatic planar graph on n vertices with maximum degree at most Δ can be embedded into every graph on n vertices with minimum degree at least $(\frac{2}{3} + \gamma)n$.*
(b) *Every planar graph on n vertices with maximum degree at most Δ can be embedded into every graph on n vertices with minimum degree at least $(\frac{3}{4} + \gamma)n$.*

This extends a result by Kühn, Osthus, and Taraz [22], who proved that for every graph H with minimum degree at least $(\frac{2}{3} + \gamma)n$ *there exists a particular* spanning triangulation G that can be embedded into H. Using Corollary 12 it is moreover possible to formulate corresponding generalizations for graphs of fixed genus and for H-minor free graphs for any fixed H.

3. Proofs

3.1. Separation and bandwidth

For the proof of Theorem 6 we will use the following decomposition result which roughly states the following. If the removal of a small separator S decomposes the vertex set of a graph G into relatively small components $R_i \dot\cup P_i$ such that the vertices in P_i form a "buffer" between the vertices in the separator S and the set of remaining vertices R_i in the sense that $\mathrm{dist}_G(S, R_i)$ is sufficiently big, then the bandwidth of G is small.

Lemma 16 (decomposition lemma). *Let $G = (V, E)$ be a graph and S, P, and R be vertex sets such that $V = S \dot\cup P \dot\cup R$. For $b, r \in \mathbb{N}$ with $b \geq 3$ assume further that there are decompositions $P = P_1 \dot\cup \ldots \dot\cup P_b$ and $R = R_1 \dot\cup \ldots \dot\cup R_b$ of P and R, respectively, such that the following properties are satisfied:*

(i) $|R_i| \leq r$,
(ii) $e(R_i \dot\cup P_i, R_j \dot\cup P_j) = 0$ *for all* $1 \leq i < j \leq b$,
(iii) $\mathrm{dist}_G(u, v) \geq b/2$ *for all* $u \in S$ *and* $v \in R_i$ *with* $i \in [b]$.

Then $\mathrm{bw}(G) \leq 2(|S| + |P| + r)$.

Proof. Assume we have $G = (V, E)$, $V = S \dot\cup P \dot\cup R$ and $b, r \in \mathbb{N}$ with the properties stated above. Our first goal is to partition V into pairwise disjoint sets $B_1, \ldots, B_b$, which we call *buckets*, and that satisfy the following property:

$$\text{If } \{u, v\} \in E \text{ for } u \in B_i \text{ and } v \in B_j \text{ then } |i - j| \leq 1. \tag{1}$$

To this end all vertices of R_i are placed into bucket B_i for each $i \in [b]$ and the vertices of S are placed into bucket $B_{\lceil b/2 \rceil}$. The remaining vertices from the sets P_i are distributed over the buckets according to their distance from S: vertex $v \in P_i$ is assigned to bucket $B_{j(v)}$ where $j(v) \in [b]$ is defined by

$$j(v) := \begin{cases} i & \text{if } \mathrm{dist}(S, v) \geq |b/2 - i|, \\ \lceil b/2 \rceil - \mathrm{dist}(S, v) & \text{if } \mathrm{dist}(S, v) < b/2 - i \\ \lceil b/2 \rceil + \mathrm{dist}(S, v) & \text{if } \mathrm{dist}(S, v) < i - b/2. \end{cases} \tag{2}$$

This placement obviously satisfies

$$|B_j| \leq |S| + |P| + |R_i| \leq |S| + |P| + r \tag{3}$$

by construction and condition (i). Moreover, we claim that it guarantees condition (1). Indeed, let $\{u, v\} \in E$ be an edge. If u and v are both in S then clearly (1) is satisfied. Thus it remains to consider the case where, without loss of generality, $u \in R_i \dot\cup P_i$ for some $i \in [b]$. By condition (ii) this implies $v \in S \dot\cup R_i \dot\cup P_i$. First assume that $v \in S$. Thus $\mathrm{dist}(u, S) = 1$ and from condition (iii) we infer that $u \in P_i$. Accordingly u is placed into bucket $B_{j(u)} \in \{B_{\lceil b/2\rceil-1}, B_{\lceil b/2\rceil}, B_{\lceil b/2\rceil+1}\}$ by (2) and v is placed into bucket $B_{\lceil b/2\rceil}$ and so we also get (1) in this case. If both $u, v \in R_i \dot\cup P_i$, on the other hand, we are clearly done if $u, v \in R_i$. So assume without loss of generality, that $u \in P_i$. If $v \in P_i$ then we conclude from $|\mathrm{dist}(S, u) - \mathrm{dist}(S, v)| \leq 1$ and (2) that u is placed into bucket $B_{j(u)}$ and v into $B_{j(v)}$ with $|j(u) - j(v)| \leq 1$. If $v \in R_i$, finally, observe that $|\mathrm{dist}(S, u) - \mathrm{dist}(S, v)| \leq 1$ together with condition (iii) implies that $\mathrm{dist}(S, u) \geq b/2-1$ and so u is placed into bucket $B_{j(u)}$ with $j(u) \in \{i(v), i(v)-1\}$ by (2) and v is assigned to B_i with $i = i(v)$. Thus we also get (1) in this last case.

Now we are ready to construct an ordering of V respecting the desired bandwidth bound. We start with the vertices in bucket B_1, order them arbitrarily, proceed to the vertices in bucket B_2, oder them arbitrarily, and so on, up to bucket B_b. By condition (1) this gives an ordering with bandwidth at most twice as large as the largest bucket and thus we conclude from (3) that $\mathrm{bw}(G) \leq 2(|S| + |P| + r)$.

A decomposition of the vertices of G into buckets as in the proof of Lemma 16 is also called a path partition of G and appears e.g. in [11].

Before we get to the proof of Theorem 6, we will establish the following technical observation about labelled trees.

Proposition 17. *Let b be a positive real, $T = (V, E)$ be a tree with $|V| \geq 3$, and $\ell : V \to [0, 1]$ be a real valued labelling of its vertices such that $\sum_{v\in V} \ell(v) \leq 1$. Denote further for all $v \in V$ by $L(v)$ the set of leaves that are adjacent to v and suppose that $\ell(v) + \sum_{u\in L(v)} \ell(u) \geq |L(v)|/b$. Then T has at most b leaves in total.*

Proof. Let $L \subseteq V$ be the set of leaves of T and $I := V \setminus L$ be the set of internal vertices. Clearly

$$1 \geq \sum_{v\in V} \ell(v) = \sum_{v\in I} \left(\ell(v) + \sum_{u\in L(v)} \ell(u) \right) \geq \sum_{v\in I} \frac{|L(v)|}{b} = \frac{|L|}{b}$$

which implies the assertion.

The idea of the proof of Theorem 6 is to repeatedly extract separators from G and the pieces that result from the removal of such separators. We denote the union of these separators by S, put all remaining vertices with small distance from S into sets P_i, and all other vertices into sets R_i. Then we can apply the decomposition lemma (Lemma 16) to these sets S, P_i, and R_i. This, together with some technical calculations, will give the desired bandwidth bound for G.

Proof (of Theorem 6). Let $G = (V, E)$ be a graph on n vertices with maximum degree at most $\Delta \geq 4$. Observe that the desired bandwidth bound is trivial if $\log_\Delta n - \log_\Delta \mathrm{s}(G) \leq 6$, so assume in the following that $\log_\Delta n - \log_\Delta \mathrm{s}(G) > 6$. Define

$$\beta := \log_\Delta n - \log_\Delta \mathrm{s}(G) \qquad \text{and} \qquad b := \lceil \beta \rceil \geq 7 \tag{4}$$

and observe that with this choice of β our aim is to show that $\mathrm{bw}(G) \leq 6n/\beta$.

The goal is to construct a partition $V = S \,\dot\cup\, P \,\dot\cup\, R$ with the properties required by Lemma 16. For this purpose we will recursively use the fact that G and its subgraphs have separators of size at most $\mathrm{s}(G)$. In the i-th round we will identify separators $S_{i,k}$ in G whose removal splits G into parts $V_{i,1}, \dots, V_{i,b_i}$. The details are as follows.

In the first round let $S_{1,1}$ be an arbitrary $(\mathrm{s}(G), 2/3)$-separator in G that separates G into $V_{1,1}$ and $V_{1,2}$ and set $b_1 := 2$. In the i-th round, $i > 1$, consider each of the sets $V_{i-1,j}$ with $j \in [b_{i-1}]$. If $|V_{i-1,j}| \leq 2n/b$ then let $V_{i,j'} := V_{i-1,j}$, otherwise choose an $(\mathrm{s}(G), 2/3)$-separator $S_{i,k}$ that separates $G[V_{i-1,j}]$ into sets $V_{i,j'}$ and $V_{i,j'+1}$ (where k and j' are appropriate indices, for simplicity we do not specify them further). Let S_i denote the union of all separators constructed in this way (and in this round). This finishes the i-th round. We stop this procedure as soon as all sets $V_{i,j'}$ have size at most $2n/b$ and denote the corresponding i by i^*. Then b_{i^*} is the number of sets $V_{i^*,j'}$ we end up with in the last iteration. Let further x_S be the number of separators $S_{i,k}$ extracted from G during this process in total.

Claim 18. *We have $b_{i^*} \leq b$ and $x_S \leq b - 1$.*

We will postpone the proof of this fact and first show how it implies the theorem. Set $S := \bigcup_{i \in [i^*]} S_i$, for $j \in [b_{i^*}]$ define

$$P_j := \{v \in V_{i^*,j} : \mathrm{dist}(v, S) < \beta/2\} \qquad \text{and} \qquad R_j = V_{i^*,j} \setminus P_j,$$

set $P_j = R_j = \emptyset$ for $b_{i^*} < j \le b$ and finally define $P := \bigcup_{j\in[b]} P_j$ and $R := \bigcup_{j\in[b]} R_j$.

We claim that $V = S \,\dot\cup\, P \,\dot\cup\, R$ is a partition that satisfies the requirements of the decomposition lemma (Lemma 16) with parameter b and $r = 2n/b$. To check this, observe first that for all $i \in [i^*]$ and $j, j' \in [b_i]$ we have $e(V_{i,j}, V_{i,j'}) = 0$ since $V_{i,j}$ and $V_{i,j'}$ were separated by some $S_{i',k}$. It follows that $e(R_j \,\dot\cup\, P_j, R_{j'} \,\dot\cup\, P_{j'}) = e(V_{i^*,j}, V_{i^*,j'}) = 0$ for all $j, j' \in [b_{i^*}]$. Trivially $e(R_j \,\dot\cup\, P_j, R_{j'} \,\dot\cup\, P_{j'}) = 0$ for all $j \in [b]$ and $b_{i^*} < j' \le b$ and therefore we get condition (ii) of Lemma 16. Moreover, condition (iii) is satisfied by the definition of the sets P_j and R_j above. To verify condition (i) note that $|R_j| \le |V_{i^*,j}| \le 2n/b = r$ for all $j \in [b_{i^*}]$ by the choice of i^* and $|R_j| = 0$ for all $b_{i^*} < j \le b$. Accordingly we can apply Lemma 16 and infer that

$$\mathrm{bw}(G) \le 2\left(|S| + |P| + \frac{2n}{b}\right). \tag{5}$$

In order to establish the desired bound on the bandwidth, we thus need to show that $|S| + |P| \le n/\beta$. We first estimate the size of S. By Claim 18 in total at most $x_S \le b-1$ separators have been extracted in total, which implies

$$|S| \le x_S \cdot \mathrm{s}(G) \le (b-1)\,\mathrm{s}(G). \tag{6}$$

Furthermore all vertices $v \in P$ satisfy $\mathrm{dist}_G(v, S) < \beta/2$ by definition. As G has maximum degree Δ there are at most $|S|(\Delta^{\beta/2} - 1)/(\Delta - 1)$ vertices $v \in V \setminus S$ with this property and hence

$$\begin{aligned}|S| + |P| &\le |S|(1 + \frac{\Delta^{\beta/2} - 1}{\Delta - 1}) \le |S|\frac{\Delta^{\beta/2}}{\Delta - 2} \le \frac{(b-1)\,\mathrm{s}(G)}{(\Delta - 2)}\sqrt{\frac{n}{\mathrm{s}(G)}} \\ &= \frac{(b-1)n}{(\Delta-2)}\sqrt{\frac{\mathrm{s}(G)}{n}}\end{aligned}$$

where the third inequality follows from (4) and (6). It is easy to verify that for any $x \ge 1$ and $\Delta \ge 4$ we have $(\Delta-2)\sqrt{x} \ge \log^2_\Delta x$. This together with (4) gives $(\Delta - 2)\sqrt{n/\mathrm{s}(G)} \ge \beta^2$ and hence we get

$$|S| + |P| \le \frac{(b-1)n}{\beta^2} \le \frac{n}{\beta}.$$

Together with (5) this yields the assertion of the theorem.

It remains to prove Claim 18. Notice that the process of repeatedly separating G and its subgraphs can be seen as a binary tree T on vertex set W whose internal nodes represent the extraction of a separator $S_{i,k}$ for some i (and thus the separation of a subgraph of G into two sets $V_{i,j}$ and $V_{i,j'}$) and whose leaves represent the sets $V_{i,j}$ that are of size at most $2n/b$. Clearly the number of leaves of T is b_{i^*} and the number of internal nodes x_S. As T is a binary tree we conclude $x_S = b_{i^*} - 1$ and thus it suffices to show that T has at most b leaves in order to establish the claim. To this end we would like to apply Proposition 17. Label an internal node of T that represents a separator $S_{i,k}$ with $|S_{i,k}|/n$, a leaf representing $V_{i,j}$ with $|V_{i,j}|/n$ and denote the resulting labelling by ℓ. Clearly we have $\sum_{w\in W} \ell(w) = 1$. Moreover we claim that

$$\ell(w) + \sum_{u\in L(w)} \ell(w) \geq |L(w)|/b \qquad \text{for all } w \in W \tag{7}$$

where $L(w)$ denotes the set of leaves that are children of w. Indeed, let $w \in W$, notice that $|L(w)| \leq 2$ as T is a binary tree, and let u and u' be the two children of w. If $|L(w)| = 0$ we are done. If $|L(w)| > 0$ then w represents a $(2/3, s(G))$-separator $S(w) := S_{i-1,k}$ that separated a graph $G[V(w)]$ with $V(w) := V_{i-1,j} \geq 2n/b$ into two sets $U(w) := V_{i,j'}$ and $U'(w) := V_{i,j'+1}$ such that $|U(w)| + |U'(w)| + |S(w)| = |V(w)|$. In the case that $|L(w)| = 2$ this implies

$$\ell(w) + \ell(u) + \ell(u') = \frac{|S(w)| + |U(w)| + |U'(w)|}{n} = \frac{|V(w)|}{n} \geq 2/b$$

and thus we get (7). If $|L(w)| = 1$ on the other hand then, without loss of generality, u is a leaf of T and $|U'(w)| > 2n/b$. Since $S(w)$ is a $(2/3, s(G))$-separator however we know that $|V(w)| \geq \frac{3}{2}|U'(w)|$ and hence

$$\begin{aligned}\ell(w) + \ell(u) = \frac{|S(w)| + |U(w)|}{n} &= \frac{|S(w)| + |V(w)| - |U'(w)| - |S(w)|}{n}\\ &\geq \frac{\frac{3}{2}|U'(w)| - |U'(w)|}{n} \geq \frac{\frac{1}{2}(2n/b)}{n}\end{aligned}$$

which also gives (7) in this case. Therefore we can apply Proposition 17 and infer that T has at most b leaves as claimed.

3.2. Boundedness

In this section we study the relation between boundedness, bandwidth and treewidth. We first give a proof of Proposition 8.

Proof (of Proposition 8). We have to show that for every graph G and every $\varepsilon > 0$ the inequality $b_\varepsilon(G) \leq 2\,\mathrm{bw}(G)/\varepsilon$ holds. Suppose that G has n vertices and let $\sigma : V \to [n]$ be an arbitrary labelling of G. Furthermore assume that $V' \subseteq V$ with $|V'| = b_\varepsilon(G)$ induces an ε-expander in G. Define $V^* \subseteq V'$ to be the first $b_\varepsilon(G)/2 = |V'|/2$ vertices of V' with respect to the ordering σ. Since V' induces an ε-expander in G there must be at least $\varepsilon b_\varepsilon(G)/2$ vertices in $N^* := N(V^*) \cap V'$. Let u be the vertex in N^* with maximal $\sigma(u)$ and $v \in V^* \cap N(u)$. As $u \notin V^*$ and $\sigma(u') > \sigma(v')$ for all $u' \in N^*$ and $v' \in V^*$ by the choice of V^* we have $|\sigma(u) - \sigma(v)| \geq |N^*| \geq \varepsilon b_\varepsilon(G)/2$. Since this is true for every labelling σ we can deduce that $b_\varepsilon(G) \leq 2\,\mathrm{bw}(G)/\varepsilon$.

The remainder of this section is devoted to the proof of Theorem 7. We will use the following lemma which establishes a relation between boundedness and certain separators.

Lemma 19 (bounded→separator). *Let G be a graph on n vertices and let $\varepsilon > 0$. If G is $(n/2, \varepsilon)$-bounded then G has a $(2\varepsilon n/3, 2/3)$-separator.*

Proof (of Lemma 19). Let $G = (V, E)$ with $|V| = n$ be $(n/2, \varepsilon)$-bounded for $\varepsilon > 0$. It follows that every subset $V' \subseteq V$ with $|V'| \geq n/2$ induces a subgraph $G' \subseteq G$ with the following property: there is $W \subseteq V'$ such that $|W| \leq |V'|/2$ and $|N_{G'}(W)| \leq \varepsilon |W|$. We use this fact to construct a $(2\varepsilon n/3, 2/3)$-separator in the following way:

1. Define $V_1 := V$ and $i := 1$.
2. Let $G_i := G[V_i]$.
3. Find a subset $W_i \subseteq V_i$ with $|W_i| \leq |V_i|/2$ and $|N_{G_i}(W_i)| \leq \varepsilon |W_i|$.
4. Set $S_i := N_{G_i}(W_i)$, $V_{i+1} := V_i \setminus (W_i \cup S_i)$.
5. If $|V_{i+1}| \geq \frac{2}{3}n$ then set $i := i + 1$ and go to step (2).
6. Set $i^* := i$ and return

$$A := \bigcup_{i=1}^{i^*} W_i, \quad B := V_{i^*+1}, \quad S := \bigcup_{i=1}^{i^*} S_i.$$

This construction obviously returns a partition $V = A \dot\cup B \dot\cup S$ with $|B| < \frac{2}{3}n$. Moreover, $|V_{i^*}| \geq \frac{2}{3}n$ and $|W_{i^*}| \leq |V_{i^*}|/2$ and hence

$$|A| = n - |B| - |S| = n - |V_{i^*+1}| - |S| =$$
$$n - (|V_{i^*}| - |W_{i^*}| - |S_{i^*}|) - |S| \leq n - \frac{|V_{i^*}|}{2} \leq \frac{2}{3}n.$$

The upper bound on $|S|$ follows easily since

$$|S| = \sum_{i=1}^{i^*} |N_{G_i}(W_i)| \leq \sum_{i=1}^{i^*} \varepsilon |W_i| = \varepsilon |A| \leq \frac{2}{3}\varepsilon n.$$

It remains to show that S separates G. This is indeed the case as $N_G(A) \subseteq S$ by construction and thus $E(A, B) = \emptyset$.

Now we can prove Theorem 7. As remarked earlier, Grohe and Marx [14] independently gave a proof of an equivalent result which employs similar ideas but doesn't use separators explicitly.

Proof (of Theorem 7). Let $G = (V, E)$ be a graph on n vertices, $\varepsilon > 0$, and let $b \geq b_\varepsilon(G)$. It follows immediately from the definition of boundedness that every subgraph $G' \subseteq G$ with $G' = (V', E')$ and $|V'| \geq 2b$ also has $b_\varepsilon(G') \leq b$.

We now prove Theorem 7 by induction on the size of G. $\mathrm{tw}(G) \leq 2\varepsilon n + 2b$ trivially holds if $n \leq 2b$. So let G have $n > 2b$ vertices and assume that the theorem holds for all graphs with less than n vertices. Then G is (b, ε)-bounded and thus has a $(2\varepsilon n/3, 2/3)$-separator S by Lemma 19. Assume that S separates G into the two subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Let $(\mathcal{X}_1, T_1)$ and $(\mathcal{X}_2, T_2)$ be tree decompositions of G_1 and G_2, respectively, and assume that $\mathcal{X}_1 \cap \mathcal{X}_2 = \emptyset$. We use them to construct a tree decomposition $(\mathcal{X}, T)$ of G as follows. Let $\mathcal{X} = \{X_i \cup S : X_i \in \mathcal{X}_1\} \cup \{X_i \cup S : X_i \in \mathcal{X}_2\}$ and $T = (I_1 \cup I_2, F = F_1 \cup F_2 \cup \{e\})$ where e is an arbitrary edge between the two trees. This is indeed a tree decomposition of G: Every vertex $v \in V$ belongs to at least one $X_i \in \mathcal{X}$ and for every edge $\{v, w\} \in E$ there exists $i \in I$ (where I is the index set of $\mathcal{X}$) with $\{v, w\} \subseteq X_i$. This is trivial for $\{v, w\} \subseteq V_i$ and follows from the definition of $\mathcal{X}$ for $v \in S$ and $w \in V_i$. Since S separates G there are no edges $\{v, w\}$ with $v \in V_1$ and $w \in V_2$. For the same reason the third property of a tree decomposition holds: if j lies on the path from i to k in T, then $X_i \cap X_k \subseteq X_j$ as the intersection is S if X_i, X_k are subsets of V_1 and V_2 respectively.

We have seen that $(\mathcal{X}, T)$ is a tree decomposition of G and can estimate its width. This gives $\mathrm{tw}(G) \leq \max\{\mathrm{tw}(G_1), \mathrm{tw}(G_2)\} + |S|$. With the induction hypothesis we have

$$\begin{aligned} \mathrm{tw}(G) &\leq \max\{2\varepsilon \cdot |V_1| + 2b,\ 2\varepsilon \cdot |V_2| + 2b\} + |S| \\ &\leq 2\varepsilon n + 2b. \end{aligned}$$

where second inequality is due to $|V_i| \leq (2/3)n$ and $|S| \leq (2\varepsilon n)/3$.

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