Problem Set 1

Exercise 1.1 (`Polymake` and H- and V-polytopes)
Experiment with the online demo of *Polymake* `http://www.polymake.org/doku.php/boxdoc`, which can convert between descriptions of polytopes as V-polytopes and as H-polytopes.

In particular, determine the vertices of the polytope in $\mathbb{R}^2$ described by the following inequalities:

$$\begin{array}{lcl} x_1 + 2x_2 & \leq & 120, \\ x_1 + x_2 & \leq & 70, \\ 2x_1 + x_2 & \leq & 100, \\ x_1 & \geq & 0, \\ x_2 & \geq & 0. \end{array}$$

(*Remark*: A V-polytope is the convex hull of a nonempty finite collection of points in $\mathbb{R}^n$. A nonempty subset of $\mathbb{R}^n$ of the form $\{x \in \mathbb{R}^n : Ax \leq b\}$ is called an H-polyhedron. A bounded H-polyhedron is called an H-polytope.)

Exercise 1.2 (Shortest Path Problem as Integer Linear Program)
Let $G = (V, E, c)$ be a digraph with positive arc weights $c : E \to \mathbb{N}$ and let $s, t \in V$ be two distinct nodes in the digraph. Devise an integer linear program that models the problem of finding a shortest directed s-t path in G. What is the encoding length of your ILP?

(*Remark:* The *encoding length* of a combinatorial optimization instance is the number of bits that are needed to describe the instance.)

Exercise 1.3 ($\mathbb{NP}$-completeness of the Shortest Path Problem)
The problem

> HAMILTONIAN PATH
> Given a graph $G = (V, E)$, does G contain a Hamiltonian path?

is $\mathbb{NP}$-complete. Use this result to show the $\mathbb{NP}$-completeness of

> SHORTEST PATH
> Given a graph $G = (V, E, c)$ with weights $c : E \to \mathbb{Z}$, two vertices $s, t \in V$, and an integer k. Is there an $s - t$-path of weight at most k?

(*Remark*: A Hamiltonian path is a path between two vertices of a graph that visits each vertex exactly once.)

Please turn over.

Exercise 1.4 (Matching Polytope)

The *matching polytope* $P_M(G)$ of a graph $G = (V, E)$ is $P_M(G) := \text{conv}\{\chi^M : M \text{ matching}\} \subseteq \mathbb{R}^E$. Let $P(G)$ and $P'(G)$ denote the polytopes

$$P(G) := \{x \in \mathbb{R}^E : (x_e \geq 0, e \in E) \wedge (\sum_{e \in \delta(v)} x_e \leq 1, v \in V)\}$$

and

$$P'(G) := \{x \in P(G) : \sum_{e \in E(S)} x_e \leq (|S| - 1)/2, \text{for all odd cardinality } S \subseteq V\}.$$

For bipartite graphs it can be shown that $P_M(G) = P(G)$.

Give a (small) example of a graph $G = (V, E)$ such that $P(G)$ contains a fractional vertex x^*, and show that $\sum_{e \in E(S)} x_e^* > (|S| - 1)/2$ holds for an odd cardinality $S \subseteq V$. Does $\chi^M \in P'(G)$ hold for every matching M in your example?

(*Remark:* χ^M denotes the incidence vector of the set $M \subseteq E$. In this exercise, you may use Polymake (see Exercise 1.1).)