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Problem Set 2

Notation: Let $c \in \mathbb{R}^n$, $d \in \mathbb{R}^m$, $D \in \mathbb{R}^{m \times n}$, and $X \subseteq \mathbb{R}^n$. For the primal problem

$$z^* = \min\{c^T x \ : \ Dx \geq d, \ x \in X\}, \tag{1}$$

we defined in the lectures the *Lagrangian dual* as

$$w^* = \max\{w(u) \ : \ u \geq 0\},$$

with

$$w(u) = \min\{c^T x + u^T(d - Dx) \ : \ x \in X\}.$$

Exercise 2.1 (Lagrangian relaxation vs. linear programming relaxation)
Consider the primal problem (1) for some finite set $X = \{x \in \mathbb{Z}^n \ : \ Ax \geq b\}$, where $A \in \mathbb{R}^{k \times n}$, $b \in \mathbb{R}^k$, $k \in \mathbb{N}$, and $\{x \in X \ : \ Dx \geq d\} \neq \emptyset$. Let

$$z^*_{\text{LP}} = \min\{c^T x \ : \ Ax \geq b, \ Dx \geq d, \ x \in \mathbb{R}^n\}$$

denote the optimal value of the usual linear programming relaxation for (1).

Show that the Lagrangian dual bound w^* is always at least as tight as the linear programming bound z^*_{LP}.

Exercise 2.2 (Lagrangian relaxation)
Geoffrion's theorem from the lecture states that under some assumptions we have $w^* = \max\{\min\{c^T x : Dx \geq d, \ x \in X\} \ : \ u \geq 0\} = \min\{c^T x \ : \ Dx \geq d, \ \text{conv}(X)\}$.

Consider the primal problem $\min\{-x_1 - x_2 \ : \ x_1 - x_2 \leq -1, \ -x_1 + x_2 \leq -1, \ x_1, x_2 \in \mathbb{Z}_+\}$. Compute for any $u \geq 0$ the $w(u) = \min\{c^T x + u^T(d - Dx) \ : \ x \in X\}$ of the Lagrangian subproblem using $X := \{(x_1, x_2) \in \mathbb{Z}^2_+ \ : \ -x_1 + x_2 \leq -1\}$. Compute the Lagrangian dual bound w^* and compare to $\min\{-x_1 - x_2 : x_1 - x_2 \leq -1, \ (x_1, x_2) \in \text{conv}(X)\}$. Is there a contradiction to Geoffrion's theorem?

Exercise 2.3 (Subdifferentials)
Let $C \subseteq \mathbb{R}^n$ be a convex set and $f : C \to \mathbb{R}$ a convex function on C. A vector $v(\overline{x}) \in \mathbb{R}^n$ is a *subgradient* of f in $\overline{x} \in C$ if

$$f(x) \geq f(\overline{x}) + v(\overline{x})^T(x - \overline{x}), \qquad \forall x \in C.$$

The *subdifferential*, denoted by $\partial f(\overline{x})$, is the set of all subgradients of f in $\overline{x}$.

(a) Determine the subdifferentials for the functions $f(x) = |x|$ and $g(x) = \max\{0, \frac{1}{2}(x^2 - 1)\}$ on $C = \mathbb{R}$.

(b) Prove the following: Let $C \subseteq \mathbb{R}^n$ be a convex set and $f : C \to \mathbb{R}$ a convex function on C. Then,

$$f(x^*) = \min\{f(x) \ : \ x \in C\} \quad \Leftrightarrow \quad 0 \in \partial f(x^*).$$

Please turn over.

Exercise 2.4 (Characterization of the subgradients of $w(u)$)
Prove the following:

> For $u \geq 0$ let $X^*(u) = \{x \in X \ : \ w(u) = c^T x + u^T(d - Dx)\}$ denote the set of optimal solutions of the *Lagrangian subproblem* $w(u) = \min\{c^T x + u^T(d - Dx) \ : \ x \in X\}$. Then, for each $x^* \in X^*(u)$, the vector $Dx^* - d$ is subgradient of $-w$ at u.

Exercise 2.5 (The subgradient method)
The subgradient method for minimizing a convex function $f : C \to \mathbb{R}$ over a convex set C can be summarized as follows:

(1) Let $k := 0$ and choose $x^{(0)} \in C$.

(2) Compute a *subgradient* $v(x^{(k)})$ at $x^{(k)}$.

(3) Select a positive scalar $\lambda^{(k)}$ ('*step size*').

(4) Set $x^{(k+1)} := x^{(k)} - \lambda^{(k)} v(x^{(k)})$ ('*Lagrangian multiplier update*').

(5) Go to Step (2) and increase k by 1 unless $v(x^{(k)}) = 0$ or a convergence criterion is met.

Apply the subgradient method to the Lagrangian dual of the knapsack problem

$$\begin{aligned} \min z &= 7x_1 + 3x_2 + 6x_3 \\ \text{s.t.} \quad & 2x_1 + 2x_2 + 4x_3 \geq 5 \\ & x_1, x_2, x_3 \in \{0, 1\}, \end{aligned}$$

which has been considered in the lectures. Start with $x^{(0)} = 0$, use the step size $\lambda^{(k)} = 2^{-k}$, use the slightly problem adapted Lagrangian multiplier updating rule $x^{(k+1)} := \max\{0, x^{(k)} - \lambda^{(k)} v(x^{(k)})\}$, and use as stopping criterion that at most 5 iterations should be performed.