Problem Set 4

Exercise 4.1 (Polarity)
Let $K \subseteq \mathbb{R}^d$ be a subset. Its *polar* is

$$K^* = \{y \in \mathbb{R}^d : y^T x \leq 1 \text{ for all } x \in K\}.$$

Determine the polars of the following sets.

(a) $K = \{0\}$.

(b) $K = \mathbb{R}^d$.

(c) $K = \{x \in \mathbb{R}^d : ||x||_2 \leq 1\}$.

(d) $K = \operatorname{conv}\left(\{-1,+1\}^d\right) = \left\{x \in \mathbb{R}^d : -1 \leq \xi_i \leq 1,\ \ i = 1,\dots,n\right\}$.

Exercise 4.2 (Polarity)
Let $K, L \subseteq \mathbb{R}^d$ be closed convex sets. Prove the following.

(a) $0 \in K^*$.

(b) K^* is closed and convex.

(c) $K \subseteq L$ implies $K^* \supseteq L^*$.

(d) If $0 \in K$, then $K^{**} = K$.

(e) If $0 \in K \cap L$, then $K \subseteq L$ if and only if $K^* \supseteq L^*$.

(f) K is bounded if and only if K^* has 0 in its interior.

(g) K^* is bounded if and only if K has 0 in its interior.

Exercise 4.3 (Submodularity of the rank function)
A function $f : 2^S \to \mathbb{R}$ is called *submodular* if

$$f(X \cap Y) + f(X \cup Y) \leq f(X) + f(Y) \text{ for all } X, Y \subseteq S.$$

Prove that the rank function r_M of a given matroid $M = (E, \mathcal{I})$ is submodular.
(As usual, $r_M : 2^E \to \mathbb{N}$ with $r_M(X) = \max\{|Y| : Y \subseteq X, Y \in \mathcal{I}\}$.)

Hint: Let $X \subseteq A \cup B$ with $|X| = r_M(A \cup B)$ and $Y \subseteq A \cap B$ with $|Y| = r_M(A \cap B)$. First show that you can assume that $Y \subseteq X$. Then use elementary set algebra.

Please turn over.

Exercise 4.4 (Submodular polyhedron greedy algorithm)
Let $S = \{s_1, \dots, s_n\}$ denote a finite set and $f : 2^S \to \mathbb{R}$ a submodular function with $f(\emptyset) = 0$. The *submodular polyhedron* P_f is defined as

$$P(f) = \left\{x \in \mathbb{R}^S : x(A) \leq f(A) \text{ for all } A \subseteq S\right\},$$

where, as usual, $x(A) = \sum_{a \in A} x_a$. For a non-negative weight vector $c \in \mathbb{R}^S_+$ consider the

$$\text{(LP)} \qquad \max\{c^T x : x \in P(f)\}.$$

In the lectures we have introduced the *greedy algorithm*:

> (1) Sort the elements of S such that $c(s_1) \geq c(s_2) \geq \cdots \geq c(s_n)$.
>
> (2) Let $V_0 = \emptyset$.
> For $i = 1$ to n let
> $V_i = V_{i-1} + s_i \quad \text{and} \quad x^*(s_i) = f(V_i) - f(V_{i-1})$.

By considering the dual problem

$$\min\{\sum_A f(A) y_A : \quad \sum_{A \ni s} y_A = c(s) \qquad \text{for all } s \in S, \qquad y_A \geq 0, \qquad \text{for all } A \subseteq S\}$$

and

$$y^*_A = \begin{cases} c(s_i) - c(s_{i+1}) & : \quad A = V_i, \\ 0 & : \quad \text{otherwise}, \end{cases}$$

with $c(s_{n+1}) = 0$, show that the greedy algorithm solves (LP) for all $c \in \mathbb{R}^S_+$.