Problem Set 6

Exercise 6.1 (Flow Decomposition)
The *Flow Decomposition* theorem stated in Exercise 5.6 is the following.

> **Theorem:** Let $G = (V, E)$ be a network with source s and sink t, and let f denote a flow in this network. Then there is a collection of flows $f_1, \ldots, f_k$ and a collection of $s - t$ paths $P_1, \ldots, P_k$ such that:
>
> (i) $k \leq |E|$;
>
> (ii) the flow value of f is equal to the sum of the flow values of the $f_1, \ldots, f_k$;
>
> (iii) the flow f_i, $i \in \{1, \ldots, k\}$, sends positive flow only on the edges of P_i.

(a) Prove this theorem.

(b) Show that the flows $f_1, \ldots, f_k$ in the Flow Decomposition theorem can be assumed to be integral if the flow f is integral.

Exercise 6.2 (Menger's Theorem and Flow Decomposition)
Menger's theorem for directed graphs is the following.

> **Theorem:** Let G be a directed graph, let s and t be two vertices, and $k \in \mathbb{N}$. Then there are k arc-disjoint $s - t$ paths if and only if after deleting any $k - 1$ arcs t is still reachable from s.

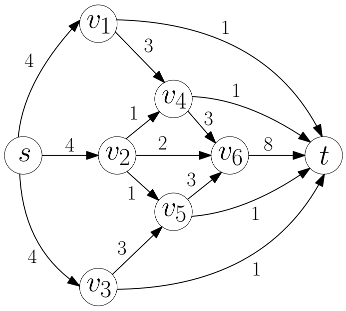
Prove this theorem.

Exercise 6.3 (Max Robust Flow)
Consider the Max Robust Flow problem where at most $k = 1$ arcs can fail in the network. Let f denote a flow in this network ($|f|$ denoting the flow value). Show that the robust value $\text{val}_r(f)$ satisfies

$$\text{val}_r(f) = |f| - \max\{f_e : e \in E\}.$$

Please turn over.

Exercise 6.4 (Max Robust Flow)
Consider the Max Robust Flow problem for the following flow network where the labeling of the arcs indicates a flow.



Show that there are two path-decompositions of this flow that yield different robust values (assuming that up to $k = 2$ arcs in this network can fail).