Network Flows: Lecture 11 (June 28, 2016)

Jannik Matuschke

1 Problem Definition

Robust network flows are a concept for dealing with uncertainty and failures in the network infrastructure. In the Maximum Robust Flow problem, we aim to find a flow that maximizes the amount of surviving flow after it is affected by the attack of an adversary, called *interdictor*, who removes a given number of arcs from the network.

The input of Maximum Robust Flow is a directed graph $D = (V, E)$ with source s, sink t and capacities $u \in \mathbb{Z}_+^E$, and an integer $k \in \mathbb{Z}_+$. Let $\mathcal{P}$ denote the set of s-t-paths and let $\mathcal{S} := \{S \subseteq E \ : \ |S| = k\}$. The goal is to find an s-t-path-flow $x \in \mathbb{R}_+^{\mathcal{P}}$ that respects the capacities u and maximizes the robust value

$$\mathrm{val_r}(x) := \sum_{P \in \mathcal{P}} x(P) - \max_{S \in \mathcal{S}} \sum_{P \in \mathcal{P} : P \cap S \neq \emptyset} x(P),$$

i.e., the remaining flow value after removal of any k arcs. This problem can also be stated as the following LP:

$$\begin{aligned}
\text{[P1]} \quad \max \quad & \sum_{P \in \mathcal{P}} x(P) - \lambda \\
\text{s.t.} \quad & \sum_{P \,:\, a \in P} x(P) \leq u(e) && \forall\, e \in E \\
& \sum_{P \,:\, P \cap S \neq \emptyset} x(P) - \lambda \leq 0 && \forall\, S \in \mathcal{S} \\
& x \geq 0
\end{aligned}$$

The dual to this problem is:

$$\begin{aligned}
\text{[D1]} \quad \min \quad & \sum_{e \in E} u(e) y(e) \\
\text{s.t.} \quad & \sum_{e \in P} y(e) + \sum_{S : P \cap S \neq \emptyset} z(S) \geq 1 && \forall\, P \in \mathcal{P} \\
& \sum_{S \in \mathcal{S}} z(S) = 1 \\
& y, z \geq 0
\end{aligned}$$

2 Hardness

It was shown by Du and Chandrasekaran [2] that the separation problem of the dual is NP-hard even when restricted to instances with $k = 2$. However, this does not imply any hardness for the problems [P1] or [D1], as the objective function of [D1] does not involve the z-variables and hence the equivalence of optimization and seperation does not apply. In the following, we discuss a new proof that shows that the problem in its general form (when k is not bounded by a constant) is *NP*-hard.

Theorem 1 ([3])**.** Maximum Robust Flow *is strongly NP-hard, even when restricted to instances where* $u(e) \in \{1, \infty\}$ *for all* $e \in E$ *and where the number of paths is polynomial in the size of the graph.*

Proof. We show this by a reduction from Clique: Given a graph $G' = (V', E')$ and $k' \in \mathbb{Z}_+$, is there a clique of size k' in G'? We construct an instance of Maximum Robust Flow consisting of a graph $G = (V, E)$, source s, sink t, capacities $u : E \to \mathbb{R}_+$, and $k \in \mathbb{Z}_+$ from the Clique instance as follows. Let

$$\ell := |V'| + 2|E'|, \quad k := k'\ell + (|V'| - k') + 2|E'|, \quad \varepsilon := \frac{1}{\ell}, \quad M := (1 + \varepsilon)k.$$

For every vertex $v \in V'$ we introduce a node a_v and two additional groups of ℓ nodes each, $A_v = \{a_{v,1}, \ldots, a_{v,\ell}\}$ and $B_v = \{b_{v,1}, \ldots, b_{v,\ell}\}$. We connect a_v to every node in B_v by an arc of capacity M, and we also connect each node $a_{v,i}$ to $b_{v,i}$ by an arc of capacity 1. For every edge $e = \{u, v\} \in E'$ we introduce two nodes a'_e, a''_e and arcs $(a'_e, b_{u,i}), (a''_e, b_{u,i}), (a'_e, b_{v,i}), (a''_e, b_{v,i})$ for $i \in \{1, \ldots, \ell\}$, each of capacity M. We denote

$$A := \bigcup_{v \in V'} (\{a_v\} \cup A_v) \cup \bigcup_{e \in E'} \{a'_e, a''_e\} \text{ and } B := \bigcup_{v \in V'} B_v.$$

We also introduce a source s and a sink t and arcs (s, a) for every $a \in A$ and (b, t) for every $b \in B$, all of infinite capacity. We then add k parallel s-t-arcs $e_1, \ldots, e_k$. Defining $h := 2 \cdot \binom{k'}{2} - 2$, we set the capacity of $e_1, \ldots, e_h$ to $1 + \varepsilon$ and the capacity of $e_{h+1}, \ldots, e_k$ to 1. We finally add two additional nodes v', v'', together with two arcs s-v'-arcs e'_1, e'_2, two v''-t-arcs e''_1, e''_2, and arcs $(s, v''), (v', t), (v', v'')$. We set the capacities $u(e'_1) = u(e''_1) = 1$, $u(e'_2) = u(e''_2) = u(v', v'') = \varepsilon$ and $u(s, v'') = u(v', t) = 1 + \varepsilon$. We let E_H denote the arcs in the subgraph H induced by the node set $\{s, v', v'', t\}$.

We now prove the following lemma, which implies Theorem 1. For convenience we will use the notation $x(e) := \sum_{P : e \in P} x(P)$ for the total flow through an arc e.

Lemma 2. *Let* (x^*, λ^*) *be an optimal solution to [P1]. Then there is a clique of size* k' *in* G' *if and only if* $x^*(v', v'') > 0$.

In order to prove Lemma 2 we first observe that, without loss of generality, we can assume that all arcs in $E \cap (A \times B)$ and the arcs $e_1, \ldots, e_k$ are saturated

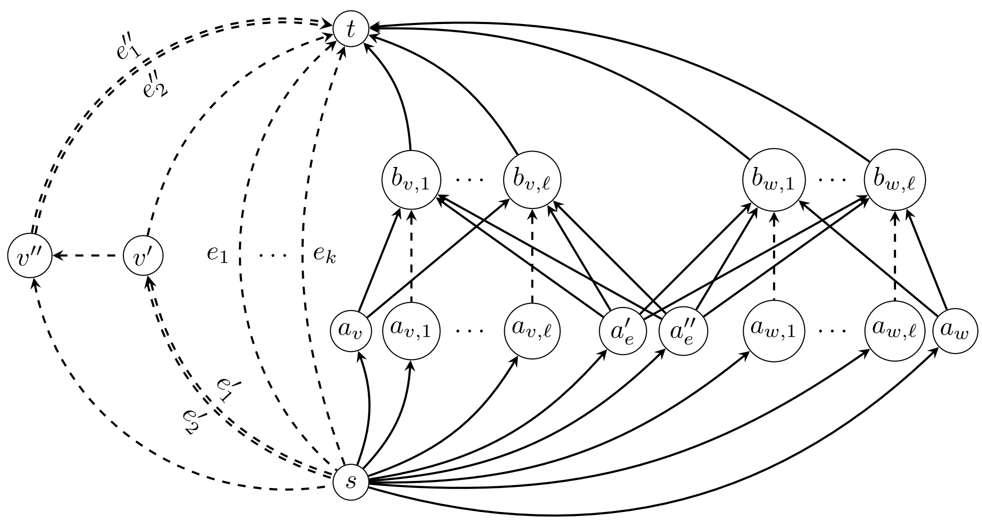

Figure 1: Construction of the reduction from Clique for two vertices v, w and an edge $e = \{v, w\}$. Solid arcs have a "large" capacity (i.e., $u(e) \in \{M, \infty\}$), dashed arcs have a "small" capacity (i.e., $u(e) \in \{\varepsilon, 1, 1 + \varepsilon\}$).

by x^*: If any of these arcs is not saturated, we can increase the flow along the unique path containing that arc and increase λ^* by the same value, not decreasing the value of the solution and not changing the flow on (v', v'').

Consider the set $F := \{e_1, \ldots, e_k\} \cup E_H$ and define

$$f_{x^*}(r) := \max\left\{ \sum_{e \in F'} x^*(e) \,:\, F' \subseteq F,\, |F'| \leq r \right\}$$

for $r \in \mathbb{N}$. We derive the following lemma.

Lemma 3. *Let* $h^* := \max\{|E'[U]| \,:\, U \subseteq V',\ |U| \leq k'\}$. *Then*

$$\lambda^* = (|V'| + 4|E'|)\ell M + k'\ell + f_{x^*}(2h^*).$$

Proof. Let $S \in \mathcal{S}$ be such that $\sum_{P \in \mathcal{P} : S \cap P \neq \emptyset} x^*(P) = \lambda^*$. Without loss of generality, we can assume that $S \cap (A \times B) = \emptyset$: If S contains an arc $(a, b) \in A \times B$, we can replace it by either of the arcs (s, a) or (b, t), each of which intersect the unique s-t-path containing (a, b).

Now define

$$U := \{v \in V' \,:\, (b, t) \in S \text{ for all } b \in B_v\}.$$

Note that $|U| \leq k'$ by choice of k and ℓ. Furthermore, note that $x^*(P) = M$ for exactly $(|V'| + 4|E'|)\ell$ paths $P \in \mathcal{P}$ by our earlier assumption and that $\lambda(S) > (|V'| + 4|E'|)\ell M$ only if S intersects all these paths by choice of M.

Therefore, we can assume that for every $v \in V'$, either $v \in U$ or $\{(s, a_v)\} \cup \{(s, a'_e), (s, a''_e) : e \in \delta(v)\} \subseteq S$. This implies that U already determines a subset S_U of

$$k_U := \ell|U| + |V'| - |U| + 2(|E'| - |E'[U]|)$$

arcs in S, destroying a flow of $(|V'| + 4|E'|)\ell M + |U|\ell$ units. The remaining $k - k_U$ arcs in S can destroy an additional flow of at most $f_{x^*}(k - k_U)$, as no arc in $E \setminus F$ carries more than 1 unit of flow after destruction of the flow paths of value M and there are at least k arcs in F with flow value at least 1. Furthermore observe that $f_{x^*}(r' + r'') \leq f_{x^*}(r') + (1+\varepsilon)r''$ as none of the edges in F carries more than $1 + \varepsilon$ units of flow. We deduce that

$$\begin{aligned}
\lambda^* \leq\ & (|V'| + 4|E'|)\ell M + |U|\ell + f_{x^*}(k - k_U) \\
=\ & (|V'| + 4|E'|)\ell M + |U|\ell + f_{x^*}((k' - |U|)(\ell - 1) + 4|E'[U]|) \\
\leq\ & (|V'| + 4|E'|)\ell M + |U|\ell + f_{x^*}(4|E'[U]|) + (1+\varepsilon)(k' - |U|)(\ell - 1) \\
=\ & (|V'| + 4|E'|)\ell M + k'\ell + (k' - |U|)(\underbrace{\varepsilon(\ell - 1) - 1}_{\leq 0}) + f_{x^*}(2\underbrace{|E'[U]|}_{\leq h^*}) \\
\leq\ & (|V'| + 4|E'|)\ell M + k'\ell + f_{x^*}(2h^*).
\end{aligned}$$

Now let $U^* \subseteq V'$ be such that $|U^*| = k'$ and $|E'[U^*]| = h^*$ and let $F^* \subseteq F$ be such that $|F^*| = 2h^*$ and $\sum_{e \in F^*} x^*(e) = f_{x^*}(2h^*)$. Consider the set

$$S^* := \bigcup_{v \in U^*} B_v \cup \{a_v : v \in V' \setminus U^*\} \cup \{a'_e, a''_e : e \notin E'[U^*]\} \cup F^*$$

and observe that $\sum_{P : P \cap S^* \neq \emptyset} x^*(P) = (|V'| + 4|E'|)\ell M + k'\ell + f_{x^*}(2h^*)$. This proves Lemma 3. □

We use Lemma 3 to prove Lemma 2 as follows. Observe that (x^*, λ^*) maximizes the quantity $\sum_{P \in \mathcal{P}} x^*(P) - \lambda^*$. As we already fixed the flow value on all paths outside of the subgraph H, we know that

$$\begin{aligned}
\sum_{P \in \mathcal{P}} x^*(P) &= \underbrace{(|V'| + 4|E'|)\ell M + \ell|V'| + \sum_{i=1}^{k} u(e_i)}_{C_1 :=} + \sum_{P \in \mathcal{P} : P \subseteq E_H} x^*(P) \\
&= C_1 + x^*(v', t) + x^*(v', v'') + x^*(s, v'')
\end{aligned}$$

where the last three summands together determine the total nominal flow through H. Defining $C_2 := (|V'| + 4|E'|)\ell M + k'\ell$, Lemma 3 states that $\lambda^* = C_2 + f_{x^*}(2h^*)$. As C_1 and C_2 do not depend on the flow in E_H, we deduce that the flow x^* in E_H maximizes the quantity

$$x^*(v', t) + x^*(v', v'') + x^*(s, v'') - f_{x^*}(2h^*).$$

First, assume G' has no clique of size k', i.e., $h^* \leq \binom{k'}{2} - 1$. In this case, $2h^* \leq h$ and hence $f_{x^*}(2h^*) = 2h^*(1+\varepsilon)$, independent of the flow values in the subgraph H, as no arc in E_H can carry more than $1+\varepsilon$ units of flow and there are already h arcs with flow value $1+\varepsilon$ in $F \setminus E_H$. Therefore, x^* maximizes $x^*(v',t) + x^*(v',v'') + x^*(s,v'')$, which implies it is the unique maximum flow in H, fulfilling $\sum_{P:(v',v'')\in P} x^*(P) = 0$.

Now assume G' has a clique of size k' and thus $h^* = \binom{k'}{2}$. In this case $2h^* = h + 2$ and hence

$$f_{x^*}(2h^*) = 2h \cdot (1+\varepsilon) + \max\{1,\, x^*(v',t)\} + \max\{1,\, x^*(s,v'')\},$$

as (v',t) and (s,v'') are the only two arcs in F outside $\{e_1,\dots,e_h\}$ that can carry more than 1 unit of flow. Thus x^* maximizes

$$x^*(v',t) + x^*(v',v'') + x^*(s,v'') - \max\{1,\, x^*(v',t)\} - \max\{1,\, x^*(s,v'')\}.$$

This term is maximized for $x^*(v',t) = x^*(s,v'') = 1$ and $x^*(v',v'') = \varepsilon$.

The above two observations conclude the proof of Lemma 2.

Note that the size of the graph $G = (V,E)$ constructed in the reduction is polynomial in the size of G'. Furthermore observe that $|\mathcal{P}| \leq |E|$ and that all capacities are polynomial in the size of G (note that the capacity ∞ can be replaced by $|E|M$). To obtain an equivalent instance in which only the capacities 1 and ∞ occur, observe that we can replace any arc of capacity u by concatenating an arc of capacity ∞ and u/ε parallel arcs of capacity ε, and scale everything by the factor ℓ. This concludes the proof of Theorem 1. □

3 Approximation

We now use a network flow problem with parametric capacities to obtain an approximation algorithm for MAXIMUM ROBUST FLOW. The results in this section were first shown by Bertsimas, Nasrabadi, and Orlin [1].

Consider the following relaxation of the dual [D1]:

$$\begin{aligned}
\text{[D2]} \quad \min \quad & \sum_{e\in E} u(e)y(e) \\
\text{s.t.} \quad & \sum_{e\in P} y(e) + z(e) \geq 1 \qquad \forall\, P \in \mathcal{P} \\
& \sum_{S\in\mathcal{S}} z(e) = k \\
& y, z \geq 0
\end{aligned}$$

and the corresponding primal program:

$$
\begin{aligned}
\text{[P2]} \quad \max \quad & \sum_{P \in \mathcal{P}} x(P) - k\lambda \\
\text{s.t.} \quad & \sum_{P\,:\,e \in P} x(P) \le u(e) \qquad \forall\, e \in E \\
& \sum_{P\,:\,e \in P} x(P) - \lambda \le 0 \qquad \forall\, e \in E \\
& x \ge 0
\end{aligned}
$$

We observe that every feasible solution to [P2] yields a feasible robust flow (i.e., a solution to [P1]) of at least the same value. Conversely, every feasible solution to [D1] can be turned into a solution to [D2].

Lemma 4.

- *If (x, λ) is a feasible solution to [P2], then $(x, k\lambda)$ is a feasible solution to [P1] of the same value.*
- *If (y, z) is a feasible solution to [D1] then (y, z') with $z'(e) := \sum_{S\,:\,e \in S} z(S)$ for $e \in E$ is a feasible solution to [D2] of the same value.*

We now construct a pair of optimal solutions for [P2] and [D2] and investigate under which conditions these solutions also yield optimal solutions for [P1] and [D1].

For $\lambda \in \mathbb{Q}_+$ define $u_\lambda(e) := \min\{u(e), \lambda\}$. Define

$$f(\lambda) := \min\{u_\lambda(C) \;:\; C \text{ is an } s\text{-}t\text{-cut}\}.$$

Observe that f is piecewise linear concave function. Let $\lambda^* := \min\{\lambda \;:\; \partial_+ f(\lambda) < k\}$ and define $L(C, \lambda) := \{e \in C \;:\; u(e) \ge \lambda\}$ and $L_+(C, \lambda) := \{e \in C \;:\; u(e) > \lambda$. Let $C', C'' \subseteq E$ be s-t-cuts with $u_{\lambda^*}(C') = u_{\lambda^*}(C'') = f(\lambda^*)$, $|\underbrace{L(C', \lambda^*)}_{A}| \ge k$, and $|\underbrace{L_+(C'', \lambda^*)}_{B}| < k$ (w.l.o.g. we assume $C' \cap C'' = \emptyset$). Define

$$
\begin{aligned}
y(e) &:= \frac{k - |B|}{|A| - |B|} \quad \forall\, e \in C' \setminus A \\
z(e) &:= \frac{k - |B|}{|A| - |B|} \quad \forall\, e \in A \\
y(e) &:= \frac{|A| - k}{|A| - |B|} \quad \forall\, e \in C'' \setminus B \\
z(e) &:= \frac{|A| - k}{|A| - |B|} \quad \forall\, e \in B
\end{aligned}
$$

and $y(e) := 0$ and $z(e) := 0$ for all $e \in E$ not specified above. Furthermore, let x^* be a maximum s-t-flow in G w.r.t. capacities u_{λ^*}.

Lemma 5. *(x, λ^*) is an optimal solution to [P2]. (y, z) is an optimal solution to [D2].*

Lemma 6. *If $|A| = k$, then $(x, k\lambda^*)$ is an optimal solution to [P1] and there exists an optimal solution (y, z') to [D1] with $\sum_{S:e\in S} z'(S) = z(e)$.*

Proof. By Lemma 4, $(x, k\lambda^*)$ is a feasible solution to [P1]. We show that that there is feasible solution (y, z') to [D1] with $\sum_{S:e\in S} z'(S) = z(e)$. Note that this immediately implies optimality of both solutions as their value is equal. Define $z'(A) := 1$ and $z'(S) := 0$ for $S \in \mathcal{S} \setminus \{A\}$. It is easy to check that $\sum_{S:e\in S} z'(S) = z(e)$ for all $e \in E$ and $z' \in \{\tilde{z} \in \mathbb{R}_+^{\mathcal{S}} : \tilde{z}(S) = 1\}$. Further note that $y(e) = 1$ for all $e \in C' \setminus A$. As every path $P \in \mathcal{P}$ has to intersect A or $C' \setminus A$, we conclude that (y, z') is a feasible solution to [D1]. □

Theorem 7. $\mathrm{OPT}([\mathrm{D2}]) \geq \frac{k+1}{k^2/4+k+1} \mathrm{OPT}([\mathrm{D1}])$.

Proof. By Lemma 6 we can assume $|A| \geq k+1$, as otherwise $\mathrm{OPT}([\mathrm{D2}]) = \mathrm{OPT}([\mathrm{D1}])$. Now consider the following solution to [D1]. For every $F \subseteq A$ with $|F| = k - |B|$ set $\bar{z}_{B\cup F} := 1/\binom{|A|}{k-|B|}$, i.e., always interdict all arcs of B and in addition interdict each arc $a \in A$ with uniform probability $(k - |B|)/|A|$. Accordingly set $\bar{y}(e) := (k - |B|)/|A|$ for all $e \in C' \setminus A$ and set $\bar{y}(e) := (|A| - k + |B|)/|A|$ for all $e \in C'' \setminus B$. Observe that $(\bar{y}, \bar{z})$ is a feasible solution to [D1] of value

$$\mathrm{val}(\bar{y}, \bar{z}) = \frac{k - |B|}{|A|} u(C' \setminus A) + \frac{|A| - k + |B|}{|A|} u(C'' \setminus B).$$

and that further

$$\mathrm{OPT}([\mathrm{D2}]) = \underbrace{\frac{k - |B|}{|A| - |B|}}_{\geq \frac{k-|B|}{|A|}} u(C' \setminus A) + \frac{|A| - k}{|A| - |B|} u(C'' \setminus B).$$

This implies that

$$\frac{\mathrm{OPT}([\mathrm{D2}])}{\mathrm{OPT}([\mathrm{D1}])} \geq \frac{\mathrm{OPT}([\mathrm{D2}])}{\mathrm{val}(\bar{y}, \bar{z})} \geq \frac{|A| - k}{|A| - |B|} \cdot \frac{|A|}{|A| - k + |B|} \geq \gamma_k^*,$$

where γ_k^* is the optimal solution value of the optimization problem

$$\begin{aligned} \min \gamma_k(a, b) \ &:= \ \frac{(a - k)a}{(a - b)(a - k + b)} \\ \text{s.t. } a &\geq k + 1 \\ k - 1 &\geq b \geq 1. \end{aligned}$$

We observe that $\gamma_k(a, b) = \frac{a(a-k)}{a(a-k)+b(k-b)}$. As $a > k > b$, the minimum is attained at when choosing $a = k + 1$ and $b = k/2$. The lemma follows from $\gamma_k^* = \gamma_k(k + 1, k/2) = \frac{k+1}{k^2/4+k+1}$. □

Remark 8. *For even k, the factor given in Theorem 7 is tight. To see this just consider a graph with three vertices s, v, t such that there are $k+1$ parallel arcs of infinite capacity from s to v, $k/2$ arcs of infinite capacity from v to t, in addition to 1 non-interdictable arc of capacity 1 from v to t.*

Corollary 9. *There is a $\frac{k+1}{k^2/4+k+1}$-approximation algorithm for [P1] and [D1].*

References

[1] D. Bertsimas, E. Nasrabadi, and J. B. Orlin. On the power of randomization in network interdiction. Technical report, arXiv:1312.3478, 2013.

[2] D. Du and R. Chandrasekaran. The maximum residual flow problem: NP-hardness with two-arc destruction. *Networks*, 50(3):181–182, 2007.

[3] Y. Disser, J. Matuschke, and G. Oriolo. The complexity of computing a robust flow. Unpublished manuscript.