Fortsetzung Derangements:

VL 6, 25.11.09

$|A| = n$, $\mathcal{S} := \{ f: A \to A,\ f \text{ bijektiv} \}$ („Permutationen")

$\mathcal{D} := \{ f \in \mathcal{S} : \forall x \in A \text{ gilt } f(x) \neq x \}$ („Derangements")

$x \in A$, $\mathcal{F}_x := \{ f \in \mathcal{S} : f(x) = x \}$

$$\Rightarrow \mathcal{D} = \underset{n!}{\mathcal{S}} \setminus \underbrace{\bigcup_{x \in A} \mathcal{F}_x}_{\text{wieviele?}}$$

Bsp: $A = [n]$, $\mathcal{F}_7 := \{ f \in \mathcal{S} : f(7) = 7 \}$. $|\mathcal{F}_7| = (n-1)!$

$|\mathcal{F}_7 \cup \mathcal{F}_{13}| \neq |\mathcal{F}_7| + |\mathcal{F}_{13}|$, denn $\mathcal{F}_7 \cap \mathcal{F}_{13} \neq \emptyset$

$\Rightarrow |\mathcal{F}_7 \cup \mathcal{F}_{13}| = |\mathcal{F}_7| + |\mathcal{F}_{13}| - |\mathcal{F}_7 \cap \mathcal{F}_{13}|$

$$\left| \bigcup_{x \in A} \mathcal{F}_x \right| \overset{4.6}{=} \sum_{r=1}^{n} \left((-1)^{r-1} \sum_{I \in \binom{A}{r}} \left| \bigcap_{x \in I} \mathcal{F}_x \right| \right)$$

und $\left| \bigcap_{x \in I} \mathcal{F}_x \right| = \left| \{ f: A \setminus I \xrightarrow{\text{bij}} A \setminus I \} \right| = (n - |I|)! = (n-r)!$

$= \{ f \in \mathcal{S} : \forall x \in I\ f(x) = x \}$

$$\Rightarrow \left| \bigcup_{x \in A} \mathcal{F}_x \right| = \sum_{r=1}^{n} \left((-1)^{r-1} \binom{n}{r} (n-r)! \right)$$

Vereinfache: $\left|\bigcup_{x\in A} \mathcal{F}_x\right|$

$$= \sum_{r=1}^{n} (-1)^{r-1}\binom{n}{r}(n-r)! = \sum_{r=1}^{n}(-1)^{r-1}\frac{n!}{r!}$$

$$= -n!\sum_{r=1}^{n}\frac{(-1)^r}{r!}$$

$$\Rightarrow |\mathcal{D}| = |\mathcal{S}| - \left|\bigcup_{x\in A}\mathcal{F}_x\right| = n! - -n!\sum_{r=1}^{n}\frac{(-1)^r}{r!}$$

$$= \underbrace{n!} + n!\left(\sum_{r=0}^{n}\frac{(-1)^r}{r!}\ \underbrace{-1}\right)$$

$$= n! \sum_{r=0}^{n}\frac{(-1)^r}{r!}$$

$$\sum_{r=0}^{n}\frac{x^r}{r!}\xrightarrow{n\to\infty} e^x$$

$x=-1$ $\qquad e^{-1}=\frac{1}{e}$

Analysis: $\displaystyle\sum_{r=0}^{n}\frac{(-1)^r}{r!}\xrightarrow{n\to\infty}\frac{1}{e}\approx 0{,}37$ $\qquad = 0{,}37\ldots$

d.h. $\approx 37\%$ aller Permutationen sind fixpunktfrei.

Satz 5.10 $P_{n,k}$ = Anzahl der ung. k-Partitionen der Zahl n.

a) $k > n \Rightarrow P_{n,k} = 0$

b) $n \geq 1 \Rightarrow P_{n,0} = 0$

c) ~~$1 \leq k \leq n$~~ $\Rightarrow P_{n+k,k} = \sum_{i=0}^{k-1} P_{n,k-i}$

Bsp: $P_{5,3} = ?$ $\quad k=3, \quad n=2$

$$P_{5,3} = \sum_{i=0}^{2} P_{2,3-i} = \underbrace{P_{2,3}}_{a) \Rightarrow 0} + \underbrace{P_{2,2}}_{2=1+1} + \underbrace{P_{2,1}}_{2=2}.$$

$+ 1 + 1 = 2$

Test: $5 = 1+2+2$

$= 1+1+3.$

III. Relationale Strukturen

6. Graphen

Knotenmenge

Def 6.1 Graph $G = (V, E)$: endliche Menge $V \neq \emptyset$

$E \subset \binom{V}{2}$ Kantenmenge.

$x, y \in V$ benachbart $:\iff \{x,y\} \in E$

$N(x) :=$ Nachbarn von x $:= \{ y \in V : \{x,y\} \in E \}$.

$\deg(x) :=$ Grad von x $:= |N(x)|$.

x ist Blatt von G $:\iff \deg(x) = 1$.

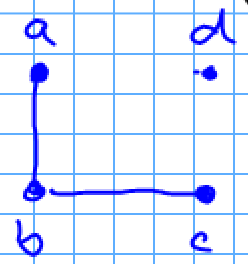
$\delta(G) :=$ Minimalgrad von G $:= \min\{ \deg(x) : x \in V \}$

$\Delta(G) :=$ Maximalgrad " $\max\{ \quad " \quad \}$

Bsp 6.2 $G = (V, E)$ mit

$V = \{a, b, c, d\}$, $E = \{ \{a,b\}, \{b,c\} \}$

$\Delta(G) = 2$



$N(b) = \{a, c\}$

$\deg(b) = 2$

a, c Blatt

$\delta(G) = 0$

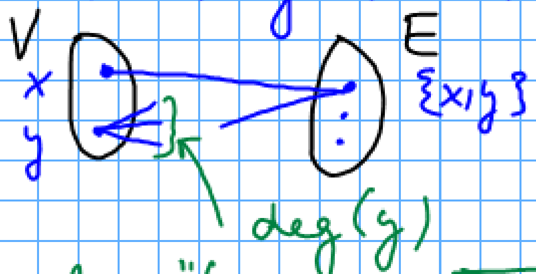
Prop 6.3 Sei $G=(V,E)$ ein Graph. Dann

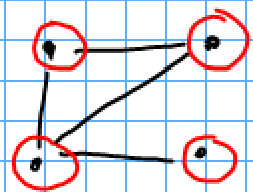
a) $\sum_{v \in V} \deg(v) = 2|E|$

b) G hat eine gerade Anzahl von Knoten ungeraden Grades.

c) Wenn $|V| \geq 2$, dann hat G mindestens zwei Knoten gleichen Grades.

Bew:

a) 



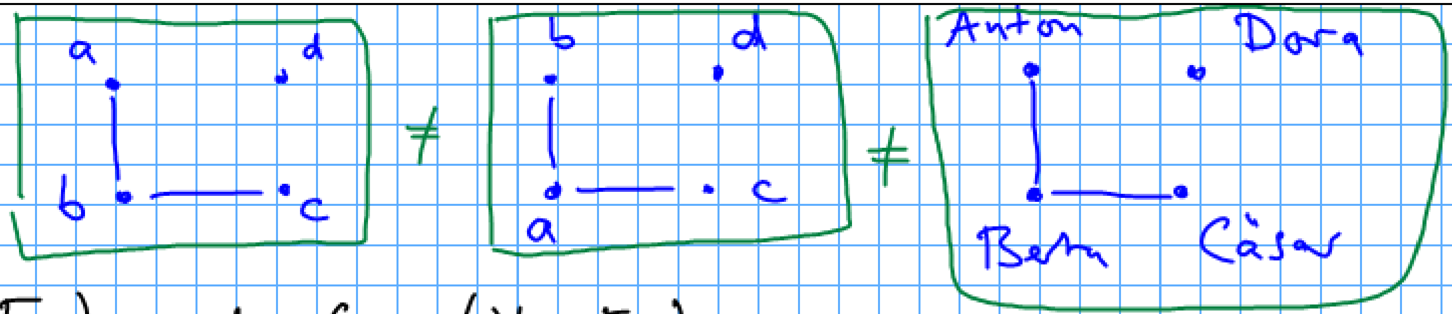
Doppeltes Abzählen: $\sum_{v \in V} \deg(v) = \sum_{e \in E} 2 = 2|E|$.

b) $\underbrace{2|E|}_{\text{gerade}} \overset{a)}{=} \sum_{v \in V} \deg(v) = \underbrace{\sum_{\substack{v \in V: \\ \deg(v) \text{ gerade}}} \deg(v)}_{\text{gerade}} + \underbrace{\sum_{\substack{v \in V: \\ \deg(v) \text{ ungerade}}} \deg(v)}_{\text{gerade}}$

$\Rightarrow$ Anzahl Sum. gerade

c) HA.

Def 6.4



$G_1 = (V_1, E_1)$ und $G_2 = (V_2, E_2)$ heißen isomorph (schreibe $G_1 \cong G_2$), wenn

$\exists$ Bijektion $f: V_1 \longrightarrow V_2$ mit

$$\{v, w\} \in E_1 \iff \{f(v), f(w)\} \in E_2 \qquad \forall v, w \in V_1.$$

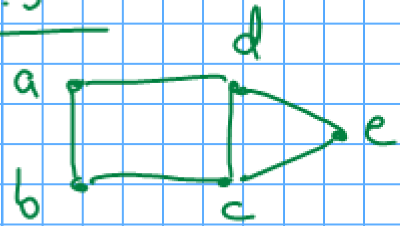
f heißt Graphenisomorphismus.

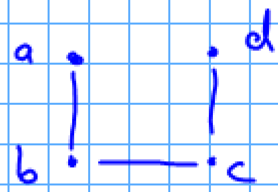
$H = (W, F)$ noch ein Graph.

$H \subset G$ $:\Longleftrightarrow$ H Subgraph von G $:\Longleftrightarrow$ $W \subset V$, $F \subset E$

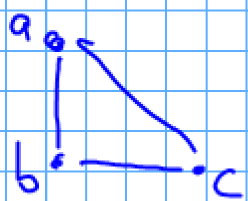
$H = G[W]$ $:\Longleftrightarrow$ H induzierter Subgraph von G $:\Longleftrightarrow$ $W \subset V$, $F = E \cap \binom{W}{2}$
(der durch W)

Bsp 6.5

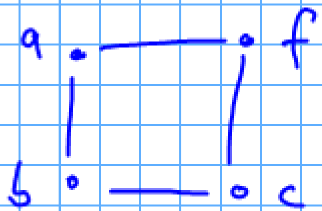
$G =$ 

$H_1 =$ 

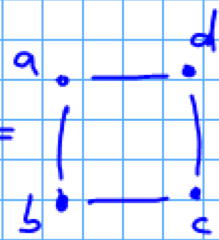
$H \subset G$, nicht induziert.

$H_2 =$ 

$H_2 \not\subset G$.

$H_3 =$ 

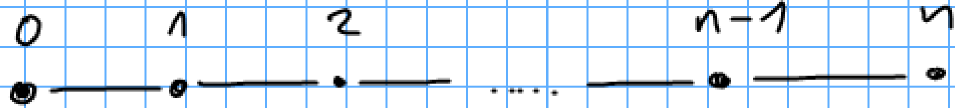
$H_3 \not\subset G$

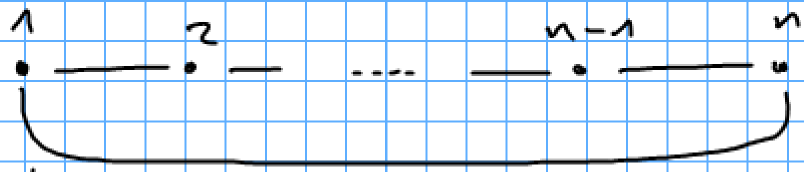
aber G hat einen Subgraphen $H_4 =$ 

und $H_4 \cong H_3$.

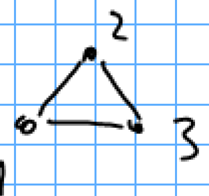
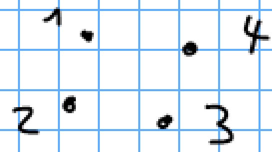
$H_4 = G[\{a, b, c, d\}]$, ist also induziert.

Def 6.6

$P_n :=$  für $n \in \mathbb{N}_0$

$C_n :=$  für $n \in \mathbb{N}$, $n \geq 3$.

$K_n := \left([n], \binom{[n]}{2} \right)$, $E_n := ([n], \emptyset)$ für $n \in \mathbb{N}$

$K_3 =$  $E_4 =$ 

Wenn $G \cong P_n$, dann heißt G Weg der Länge n.

Wenn $G =$ , dann heißen x und y Anfangs- und Endknoten, und G heißt x,y-Weg.

Wenn $G \cong C_n$, dann heißt G Kreis der Länge n.

$G \cong K_n$, Clique der Größe n

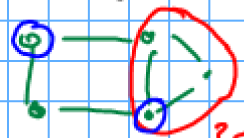
$G \cong E_n$, stabile Menge der Größe n

Cliquenzahl von G := $\omega(G) := \max\{\ell \in \mathbb{N} : \exists S \subset V \text{ mit } G[S] \cong K_\ell\}$

Stabilitätszahl von G := $\alpha(G) := \max\{\ell \in \mathbb{N} : \quad E_\ell\}$

Bsp 6.7

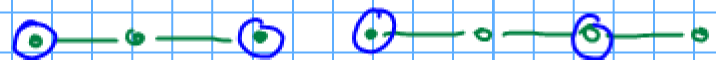
o—o— .... —o—o 2-stabile

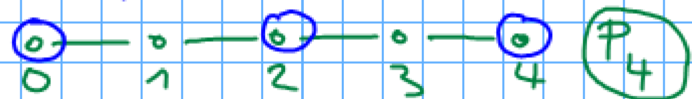


a) $\omega(K_n) = n$, $\alpha(K_n) = 1$ $\quad \forall n \in \mathbb{N}$

b) $\omega(E_n) = 1$, $\alpha(E_n) = n$ $\quad \forall n \in \mathbb{N}$

c) $\omega(P_n) = 2$, $\alpha(P_{2n}) = \alpha(P_{2n+1}) = n+1$ $\quad \forall n \in \mathbb{N}$





d) $\omega(C_3) = 3$, $\omega(C_n) = 2$ $\quad \forall n \in \mathbb{N},\ n \geq 4$

e) $\alpha(C_{2n}) = n$, $\alpha(C_{2n+1}) = n$ $\quad \forall n \in \mathbb{N},\ n \geq 2.$