VL 13, 3.2.2010

12. Fibonacci- und Catalan-Zahlen

Bsp 12.1 $F_0 := 0$, $F_1 := 1$, $F_{n+2} := F_{n+1} + F_n \quad \forall n \in \mathbb{N}_0$

$\to F_2 = F_1 + F_0 = 1 + 0 = 1$

$\to F_3 = F_2 + F_1 = 1 + 1 = 2 \quad \dots \quad F_n = ?$

① $$F(x) := \sum_{n=0}^{\infty} F_n x^n$$

② $$F(x) = F_0 + F_1 x + \sum_{n=2}^{\infty} F_n x^n$$
$$= x + \sum_{n=0}^{\infty} F_{n+2} x^{n+2}$$
$$= x + \sum_{n=0}^{\infty} (F_{n+1} + F_n) x^{n+2}$$

③ $$F(x) = x + x \sum_{n=0}^{\infty} F_{n+1} x^{n+1} + x^2 \sum_{n=0}^{\infty} F_n x^n$$
$$= x + x\left(F(x) - F_0\right) + x^2 F(x)$$
$$= x + x F(x) + x^2 F(x)$$

4 $$F(x) - x F(x) - x^2 F(x) = x$$
$$\Rightarrow F(x)(1 - x - x^2) = x$$
$$\Rightarrow F(x) = \frac{x}{1-x-x^2}.$$

5 $$\frac{x}{1-x-x^2} = \frac{\alpha}{1-\gamma x} + \frac{\beta}{1-\delta x}$$ für geeignete $\alpha, \beta, \gamma, \delta \in \mathbb{R}$

$$= \frac{\alpha(1-\delta x) + \beta(1-\gamma x)}{(1-\gamma x)(1-\delta x)}$$

Versuch: $x = \alpha(1-\delta x) + \beta(1-\gamma x)$ (1)

$$-x^2 - x + 1 = (1-\gamma x)(1-\delta x) \quad (2)$$

Nullstellen von $\uparrow$:

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\frac{1}{\gamma} = \frac{1-\sqrt{1+4}}{-2} = \frac{1-\sqrt{5}}{-2}$$
$$\Rightarrow \gamma = \frac{-2}{1-\sqrt{5}} = \frac{-2(1+\sqrt{5})}{(1-\sqrt{5})(1+\sqrt{5})} = \frac{-2-2\sqrt{5}}{1-5} = \frac{1+\sqrt{5}}{2}.$$

$$\frac{1}{\delta} = \frac{1+\sqrt{1+4}}{-2} = \frac{1+\sqrt{5}}{-2} \Rightarrow \delta = \ldots = \frac{1-\sqrt{5}}{2}$$

Setze $x := \frac{1}{\delta}$ in (1) und erhalte

$$\frac{1}{\delta} = \beta\left(1 - \gamma \cdot \frac{1}{\delta}\right) = \beta\, \frac{\delta - \gamma}{\delta}$$

$$\Rightarrow \beta = \frac{1}{\delta - \gamma} = \frac{1}{\frac{1 - \sqrt{5} - 1 - \sqrt{5}}{2}} = \frac{2}{-2\sqrt{5}} = -\frac{1}{\sqrt{5}}.$$

Setze $x := 0$ in (1) und erhalte

$$0 = \alpha + \beta \quad \Rightarrow \quad \alpha = +\frac{1}{\sqrt{5}}$$

Bem: „Vorschlag".

$$\Rightarrow F(x) \overset{(1)-(4)}{=} \frac{x}{1 - x - x^2} = \frac{\alpha}{1 - \gamma x} + \frac{\beta}{1 - \delta x}$$

$$\overset{11.5a)}{=} \alpha \sum_{n=0}^{\infty} \gamma^n x^n + \beta \sum_{n=0}^{\infty} \delta^n x^n$$

(6)

$$\sum_{n=0}^{\infty} F_n x^n = F(x) = \sum_{n=0}^{\infty} \left(\alpha \gamma^n + \beta \delta^n\right) x^n$$

$$\overset{11.6}{\Rightarrow} F_n = \alpha \gamma^n + \beta \delta^n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2}\right)^n.$$

Konvergenz von $F(x)$?

$$\frac{F_n}{F_{n+1}} = \frac{\alpha\gamma^n + \beta\delta^n}{\alpha\gamma^{n+1} + \beta\delta^{n+1}} = \frac{\cancel{\alpha}\cancel{\gamma^n}\left(1 + \frac{\beta}{\alpha}\left(\frac{\delta}{\gamma}\right)^n\right)}{\cancel{\alpha}\cancel{\gamma^{n+1}}\left(1 + \frac{\beta}{\alpha}\left(\frac{\delta}{\gamma}\right)^{n+1}\right)}$$

$\beta = -\alpha$

$$= \frac{1}{\gamma}\ \frac{1 - \left(\frac{\delta}{\gamma}\right)^n}{1 - \left(\frac{\delta}{\gamma}\right)^{n+1}}$$

wg $$\left|\frac{\delta}{\gamma}\right| = \left|\frac{\frac{1-\sqrt{5}}{2}}{\frac{1+\sqrt{5}}{2}}\right| = \left|\frac{1-\sqrt{5}}{1+\sqrt{5}}\right| < 1$$

folgt $$\lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \frac{1}{\gamma},$$

also hat $F(x)$ wg 11.3 Konvergenzradius $\frac{1}{\gamma}$.

ab 9.2. „Probeklausur" auf der Homepage

Aufgaben werden besprochen im Wiederholungskurs

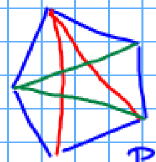
am 18.2., 10–16 Uhr, im HS 2 & 3

Klausur: 24.2., 9:30 Uhr

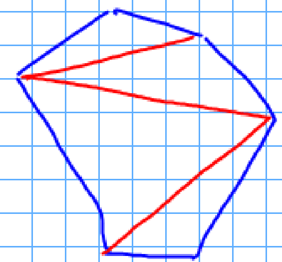
in MW 0001 / MW 1801.

Bsp 12.2

P_n = Konvexes n-Eck:

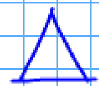


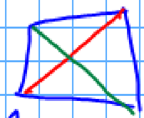
P5



P6

$C_n :=$ Anzahl Triangulierungen von P_{n+2}.

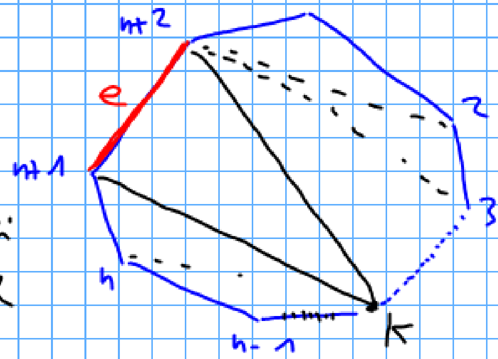




$\Rightarrow$ $C_1 = \#\text{Tri. von } P_3 = 1$

$C_2 = \#\text{Tri. von } P_4 = 2$

$k \in [n]$ $\quad C_3 = \#\text{Tri. von } P_5 = 5$



$B_k :=$ Menge aller Tri. von P_{n+2}, bei denen die Seite e ein Dreieck mit der Ecke k bildet.

$\Rightarrow$ Triangulierungen von P_{n+2} lassen sich partitionieren in $B_1 \dot\cup B_2 \dot\cup \dots \dot\cup B_n$ $\Rightarrow$ $C_n = \sum_{k=1}^{n} |B_k|$

$$|B_k| = \#\text{Tri. von } „\underbrace{1,2,3,\dots,k,n+2}_{k+1}“ \cdot \#\text{Tri. von } „\underbrace{k,k+1,\dots,n,n+1}_{\substack{n+1-(k-1)\\ = n-k+2}}“$$

$$= C_{k-1} \cdot C_{n-k}$$

$$\Rightarrow \quad C_n = \sum_{k=1}^{n} |B_k| = \sum_{k=1}^{n} C_{k-1} \cdot C_{n-k}. \qquad \forall n \in \mathbb{N}.$$

Betrachte Folge: $C_0 := 1$

$$C_n := \sum_{k=1}^{n} C_{k-1} C_{n-k} \qquad \forall n \in \mathbb{N}$$

Teste: $C_1 = C_{1-1} \cdot C_{1-1} = C_0 \cdot C_0 = 1$

$C_2 = C_{1-1} \cdot C_{2-1} + C_{2-1} \cdot C_{2-2} = 2$

$C_3 = C_{1-1} C_{3-1} + C_{2-1} C_{3-2} + C_{3-1} \cdot C_{3-3}$

$= 1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1 = 5$

Finde explizite Formel für C_n.

① $$C(x) := \sum_{n=0}^{\infty} C_n x^n$$

② $$C(x) = C_0 + \sum_{n=1}^{\infty} C_n x^n$$

$$= 1 + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} C_{k-1} C_{n-k} \right) x^n$$

(3) $$C(x) = 1 + \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} C_k \, C_{n-(k+1)} \right) x^n$$

$$= 1 + \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} C_k \, C_{n+1-(k+1)} \right) x^{n+1}$$

$$= 1 + x \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} C_k \, C_{n-k} \right) x^n$$

$$\overset{11.4\,a)}{=} 1 + x \, C(x)^2$$

(4) $$0 = x \cdot C(x)^2 - C(x) + 1$$

$$\Rightarrow \quad C(x) = \frac{+1 \pm \sqrt{1-4x}}{2x} \qquad \pm\,?$$

Wissen, dass $C(x) \xrightarrow{x \to 0} C_0 = 1$

aber $\dfrac{1 + \sqrt{1-4x}}{2x} \xrightarrow{x \to 0} \infty$,

also $$C(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

(5) $$C(x) = \frac{1 - \sqrt{1-4x}}{2x}$$

$$\Rightarrow \quad x \cdot C(x) = \frac{1}{2} - \frac{1}{2} \underbrace{(1-4x)^{\frac{1}{2}}}$$

11.8: $$(x+1)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n$$ Setze $r := \frac{1}{2}$ und $-4x$ für x

$$\Rightarrow \quad (1-4x)^{\frac{1}{2}} = \underbrace{\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-4x)^n}$$

$$\Rightarrow \quad x\,C(x) = \frac{1}{2} - \frac{1}{2} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-4)^n x^n$$

$$= -\frac{1}{2} \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} (-4)^n x^n$$

$$= -\frac{1}{2} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n+1} (-4)^{n+1} x^{n+1}$$

(6) $\sum_{n=0}^{\infty} c_n x^{n+1} = x \cdot C(x) = -\frac{1}{2} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n+1} (-4)^{n+1} x^{n+1}$

$$\Rightarrow \quad c_n = -\frac{1}{2} \binom{\frac{1}{2}}{n+1} (-4)^{n+1}$$

$$= \frac{(-4)^{n+1} \prod_{i=0}^{n} \left(\frac{1}{2} - i\right)}{(-2) \quad (n+1)!} = \frac{(-2)^{n+1} \prod_{i=0}^{n} (1-2i)}{(-2) \ (n+1)!}$$

$$= -\frac{2^n \ n! \ \boxed{\prod_{i=0}^{n} (2i-1)}}{n! \ (n+1)!} = \frac{2n \cdot (2n-2)(2n-4) \cdots 2 \cdot \boxed{(2n-1) \cdots 1 \cdot (-1)}}{n! \ (n+1) \ n!}$$

$$= \frac{(2n)!}{n! \, n!} \qquad \frac{1}{n+1}$$

$$= \frac{1}{n+1} \binom{2n}{n} .$$