last name

first name

student ID number

room

row

seat number

grade:

I hereby confirm that the exam sheet I have received is complete. I have checked the sheet and have not noticed any obvious printing error or missing pages.

student's signature

Technische Universität München
Fakultät für Mathematik
Combinatorial Optimization
Prof. Dr. Raymond Hemmecke

March 01, 2011

	I	II
1		
2		
3		
4		
5		
6		
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Please read the following instructions carefully:

- Please check the exam sheets: You should have received **6 problems** on **pages 1 to 11**, including cover sheet and summary sheet (two pages). Please check that your sheet is complete by comparison to the summary sheet.
- You have 60 minutes to complete the exam. An announcement will be made 15 minutes before examination time ends. From that point on, no one will be allowed to leave the room to minimize distraction for all participants.
- Please answer each question in the framed space immediately following the problem statement. Give precise justification for all of your answers (unless explicitly stated otherwise). Anwers will be accepted in both English and German.
- Presumably, 17 credits (out of 40) will be needed to pass the exam.
- You may tear out and keep the summary sheet at the end. Should you choose to hand in your exam prematurely you will be required to hand in *all* sheets, including the summary sheet.
- This is a closed-book examination — you are not allowed to use any utilities or devices beyond a pen during the exam! Failure to comply with these rules will result in immediate disqualification and a grade of 5.0.

for supervisory staff only:

student left the room from: ____________ to: ____________

handed in prematurely at: ____________

other remarks:

marker's signature (I)

marker's signature (II)

Problem 1 (≈ 5 credits)

I	II

Consider the knapsack problem $\max\left\{c^T x : a^T x \leq b, x \in \{0,1\}^n\right\}$. For every $\varepsilon \in (0,1)$, let $\mathcal{A}_\varepsilon$ be an algorithm with running time $n^2\varepsilon^{-2}$ that yields a feasible solution x^0 such that $c^T x^0 \geq (1-\varepsilon)c^T x^*$, where x^* is an optimal solution to the knapsack problem. Which of the following statements are true, which are false? You do *not* need to justify your answers here! *Negative credits will be given for wrong answers!*

true	**false**	
☐	☐	$\mathcal{A}_{\frac{1}{2}}$ is a $\frac{1}{2}$-approximation algorithm
☐	☐	$\mathcal{A}_{\frac{1}{2}}$ is a $\frac{1}{4}$-approximation algorithm
☐	☐	$\mathcal{A}_{\frac{1}{4}}$ is a $\frac{1}{2}$-approximation algorithm
☐	☐	$\mathcal{A}_\varepsilon$ is a PTAS
☐	☐	$\mathcal{A}_\varepsilon$ is an FPTAS

Problem 2 ($\approx$ 5 credits)

I	II

Consider the following cutting stock problem: A paper mill produces large paper rolls of width W. Before delivery to the customers, the large rolls have to be cut into smaller rolls of widths $w_1, \ldots, w_n$. The demand for each width is $d_1, \ldots, d_n$. For technological reasons, there is an additional constraint: *Each large paper roll may be cut into at most 3 smaller rolls.*

a) Below is the cutting stock formulation of the master problem discussed in the lecture. The variables μ^j, $j \in Q$, count how often a cutting pattern $x^j \in Q$ is used, Q is a set of valid cutting patterns. How would you change the primal model to account for the additional constraint? Justify your answer!

$$
\begin{aligned}
\min \sum_{j=1}^{|Q|} \mu^j& \\
\sum_{j=1}^{|Q|} x_i^j \mu^j \geq d_i& \quad \forall i = 1, \ldots, n \\
\mu^j \in \mathbb{N}_0& \quad \forall j \in Q
\end{aligned}
$$

b) For an optimal basis B of the current reduced master problem the pricing problem as discussed in the lecture turned out to be

$$
\begin{aligned}
\min 1 - y^T x^j& \\
\sum_{i=1}^{n} w_i x_i^j \leq W& \\
x^j \in \mathbb{N}_0^n&,
\end{aligned}
$$

where $y = c_B A_B^{-1}$ are the dual variables corresponding to the primal basis. How would you change the pricing problem to account for the additional constraint? Again, justify your answer!

Problem 3 (≈ 5 credits)

I	II

Let $P \subset [0,1]^4$ be the polytope defined by

$$
\begin{aligned}
x_1 + 2x_2 &\le 2 && (1)\\
x_2 + 2x_3 &\le 2 && (2)\\
x_3 + 2x_4 &\le 2 && (3)\\
2x_1 + x_4 &\le 2 && (4)\\
0 \le x &\le \mathbb{1}
\end{aligned}
$$

The point $x^* = (2/3, 2/3, 2/3, 2/3)^T$ is clearly contained in P. Apply the method of rounding to devise a new valid inequality for the integer hull P_I that shows $x^* \notin P_I$.

Problem 4 ($\approx$ 10 credits) | I | II |

Let $n \in \mathbb{N}$, $n \geq 3$, and consider the complete graph $K_n = (V, E)$ on n nodes and $m = |E|$ edges. The polytope P defined by

$$P := \operatorname{conv}\{x \in \{0,1\}^m : x_a + x_b - x_c \leq 1 \text{ for every triangle } \{a,b,c\} \subset E \text{ of } K_n\}$$

is called the *clique partitioning polytope*.

a) Prove that $\dim P = m$.

b) Show that for every triangle $\{a,b,c\} \subset E$ in K_n the inequality $x_a + x_b - x_c \leq 1$ is facet-defining for P.

c) Show that none of the inequalities $x_e \leq 1$, $e \in E$, defines a facet of P. (Hint: You may use the statement in b), even if you did not prove it.)

Consider the polytope

$$P := \operatorname{conv}\left\{x \in \{0,1\}^5 : 4x_1 + 4x_2 + x_3 + 3x_4 + 2x_5 \leq 8\right\}.$$

a) Show that the inequality

$$x_2 + x_4 + x_5 \leq 2 \tag{$*$}$$

is valid for P.

b) Devise a new valid inequality for P by lifting x_1 into inequality $(*)$.

Problem 6 (≈ 10 credits)

I	II

Let $n \in \mathbb{N}$ and consider the TRAVELING SALESMAN PROBLEM (TSP) on a complete graph $G = (V, E)$ with node set $V = \{1, \ldots, n\}$, edge set E and edge lengths $c : E \to \mathbb{N}$. For a subset $S \subset V \setminus \{1\}$ and a node $k \in S$, let $C(S, k)$ be the minimal cost of a path that starts at node 1 and then visits all the nodes in S (and only those), ending at node k.

a) For $|S| > 1$, prove the recursion formula

$$C(S, k) = \min \{C(S \setminus \{k\}, m) + c_{mk} : m \in S \setminus \{k\}\}.$$

b) Using the recursion given above, devise a dynamic programming algorithm to determine the length of a minimal traveling salesman tour in G. In particular, outline how to start the recursion, when to end the recursion and how to derive the *length* of a minimal tour. (Note that you need not specify how to construct an optimal tour!)

Problem 1

Consider the knapsack problem $\max\left\{c^Tx : a^Tx \le b, x \in \{0,1\}^n\right\}$. For every $\varepsilon \in (0,1)$, let $\mathcal{A}_\varepsilon$ be an algorithm with running time $n^2\varepsilon^{-2}$ that yields a feasible solution x^0 such that $c^Tx^0 \ge (1-\varepsilon)c^Tx^*$, where x^* is an optimal solution to the knapsack problem. Which of the following statements are true, which are false? You do *not* need to justify your answers here! *Negative credits will be given for wrong answers!*

Problem 2

Consider the following cutting stock problem: A paper mill produces large paper rolls of width W. Before delivery to the customers, the large rolls have to be cut into smaller rolls of widths $w_1, \dots, w_n$. The demand for each width is $d_1, \dots, d_n$. For technological reasons, there is an additional constraint: *Each large paper roll may be cut into at most 3 smaller rolls.*

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$$\begin{aligned} \min \sum_{j=1}^{|Q|} \mu^j & \\ \sum_{j=1}^{|Q|} x_i^j \mu^j \ge d_i & \quad \forall i = 1, \dots, n \\ \mu^j \in \mathbb{N}_0 & \quad \forall j \in Q \end{aligned}$$

b) For an optimal basis B of the current reduced master problem the pricing problem as discussed in the lecture turned out to be

$$\begin{aligned} \min\ & 1 - y^T x^j \\ & \sum_{i=1}^{n} w_i x_i^j \le W \\ & x^j \in \mathbb{N}_0^n, \end{aligned}$$

where $y = c_B A_B^{-1}$ are the dual variables corresponding to the primal basis. How would you change the pricing problem to account for the additional constraint? Again, justify your answer!

Problem 3

Let $P \subset [0,1]^4$ be the polytope defined by

$$\begin{aligned} x_1 + 2x_2 &\le 2 && (1) \\ x_2 + 2x_3 &\le 2 && (2) \\ x_3 + 2x_4 &\le 2 && (3) \\ 2x_1 + x_4 &\le 2 && (4) \\ 0 \le x &\le \mathbb{1} \end{aligned}$$

The point $x^* = (2/3, 2/3, 2/3, 2/3)^T$ is clearly contained in P. Apply the method of rounding to devise a new valid inequality for the integer hull P_I that shows $x^* \notin P_I$.

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Problem 5

Consider the polytope

$$P := \operatorname{conv}\left\{x \in \{0,1\}^5 : 4x_1 + 4x_2 + x_3 + 3x_4 + 2x_5 \le 8\right\}.$$

a) Show that the inequality

$$x_2 + x_4 + x_5 \leq 2 \qquad (*)$$

is valid for P.

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Problem 6

Let $n \in \mathbb{N}$ and consider the TRAVELING SALESMAN PROBLEM (TSP) on a complete graph $G = (V, E)$ with node set $V = \{1, \ldots, n\}$, edge set E and edge lengths $c : E \to \mathbb{N}$. For a subset $S \subset V \setminus \{1\}$ and a node $k \in S$, let $C(S, k)$ be the minimal cost of a path that starts at node 1 and then visits all the nodes in S (and only those), ending at node k.

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