Technische Universität München
Zentrum Mathematik

Combinatorial Optimization (MA 4502)

Prof. Dr. Raymond Hemmecke | Dr. Michael Ritter

Exercise Sheet 2

Exercise 2.1

Let $G = (V, E)$ be a graph with n vertices and m edges. The Maximum Matching Problem asks for a solution of the integer linear program

$$\begin{aligned} \max\ & \mathbb{1}^T x \\ & \sum_{e \in \delta(\{v\})} x_e \leq 1 \quad \text{for all } v \in V \\ & x \in \{0,1\}^m , \end{aligned}$$

where $\delta(S) := \{e \in E : |e \cap S| = |e \cap (V \setminus S)| = 1\}$ for any subset $S \subseteq V$ of vertices. The integral hull of this problem is called *matching polytope*.

Add up all inequalities of the LP relaxation of the problem above and apply rounding to derive a valid inequality for the matching polytope that is a cutting plane for the LP relaxation. Can you give a combinatorial explanation of your inequality?

Exercise 2.2

Let $G = (V, E)$ be a graph on n vertices and m edges and $c : E \to \mathbb{N}$ a cost function on the edges. The Traveling Salesman Problem asks for a solution of the integer linear program

$$\begin{aligned} \min\ & \sum_{e \in E} c_e x_e \\ & \sum_{e \in \delta(\{v\})} x_e = 2 \quad \text{for all } v \in V \qquad (1) \\ & \sum_{e \in E(S)} x_e \leq |S| - 1 \quad \text{for all } S \subseteq V, \emptyset \neq S \neq V \qquad (2) \\ & x \in \{0,1\}^m . \end{aligned}$$

The convex hull of all integer solutions P_{TSP} is called the *traveling salesman polytope*. A *comb* is a subgraph of G generated by node sets H („handle“) and $T_1, \ldots, T_t$ („teeth“) (with t odd and $t \geq 3$) such that the following holds:

i) the sets $T_1, \ldots, T_t$ are pairwise disjoint

ii) $2 \leq |T_i| \leq n - 2$ for all $i \in \{1, \ldots, t\}$

iii) for each $i \in \{1, \ldots, t\}$: $|H \cap T_i| \geq 1$ and $|T_i \setminus H| \geq 1$

a) Use the degree constraints (1) to prove that the inequality

$$2 \sum_{e \in E(H)} x_e + \sum_{i=1}^{t} \sum_{e \in \delta(H) \cap E(T_i)} x_e \leq 2\,|H| \qquad (3)$$

is valid for P_{TSP}.

Please turn over!

b) Use (3) and the subtour elimination constraints (2) to derive the valid inequality

$$\sum_{e \in E(H)} x_e + \sum_{i=1}^{t} \sum_{e \in E(T_i)} x_e \leq |H| + \frac{1}{2} \sum_{i=1}^{t} \left[(|T_i| - 1) + (|H \cap T_i| - 1) + (|T_i \setminus H| - 1) \right]. \qquad (4)$$

c) Using (4), show that the *comb inequality*

$$\sum_{e \in E(H)} x_e + \sum_{i=1}^{t} \sum_{e \in E(T_i)} x_e \leq |H| + \sum_{i=1}^{t} (|T_i| - 1) - \frac{t+1}{2}$$

is valid for the traveling salesman polytope.

Exercise 2.3

a) Imagine you had an algorithm to determine a feasible point of a polyhedron $\{x \in \mathbb{R}^n : Ax \leq b\}$ for arbitrary matrices $A \in \mathbb{R}^{m \times n}$ and vectors $b \in \mathbb{R}^m$ (if the inequalities are infeasible, the algorithm will detect and report this). How can you use that algorithm to determine the solution to a linear optimization problem?

b) Consider again the situation of a), but now with an oracle that only reports whether the given system of linear inequalities is feasible or not (instead of returning a feasible point). Use that oracle to design an algorithm that solves a linear optimization problem. Can you do this using a polynomially bounded number of calls to the oracle?

c) In this exercise we will establish a connection between three different ways to state a problem. As an example, consider the traveling salesman problem:

Problem 1: TSP – optimization problem
Instance: *$n, m \in \mathbb{N}$, a graph $G = (V, E)$ with n nodes and m edges, a weight vector $c \in \mathbb{N}^m$*
Task: *Determine if a Hamilton circuit exists in G and if so, return one with minimum total weight.*

Problem 2: TSP – function problem
Instance: *$n, m \in \mathbb{N}$, a graph $G = (V, E)$ with n nodes and m edges, a weight vector $c \in \mathbb{N}^m$*
Task: *Return the length of a Hamilton circuit with minimum total weight in G or ∞, if no Hamilton circuit exists.*

Problem 3: TSP – decision problem
Instance: *$n, m \in \mathbb{N}$, a graph $G = (V, E)$ with n nodes and m edges, a weight vector $c \in \mathbb{N}^m$, $K \in \mathbb{N}$*
Question: *Does a Hamilton circuit of total weight at most K exist in G?*

Obviously, if we have an algorithm that solves the optimization problem, the other two version can also be solved. In this exercise, we will show that the reverse is also (essentially) true.

1) Imagine you had an algorithm that could solve the decision problem for TSP which may be used as a „black box“ (more formally called an *oracle*). Design an algorithm that uses the decision oracle to solve the function problem. Can you find an algorithm that calls the oracle only polynomially often and with polynomially bounded inputs (such an algorithm would be called *oracle-polynomial*)? (In particular, you may not just call the decision oracle K times, since K is exponential in the input size of the instance – remember that K is part of the input.)

2) Imagine you had an algorithm that could solve the function problem for TSP. Design an algorithm that uses this function oracle to solve the optimization problem. Again, try to find an oracle-polynomial algorithm!