# Exercise Sheet 4

**Exercise 4.1**
Consider the polytope

$$P := \operatorname{conv}\left\{x \in \{0,1\}^7 : 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19\right\}.$$

a) Show that the inequality $x_3 + x_4 + x_5 + x_6 \leq 3$ is valid for $P$.

b) Compute liftings for the above inequality by sequentially lifting the first, second and seventh coordinate. You may use any ILP solver for your computations (try, e. g., Xpress, CPLEX or Gurobi) or compute the solutions yourself. Is the obtained inequality a facet of $P$?

**Exercise 4.2**
Let $P$ be the polyhedron in $\mathbb{R}^2$ defined by

$$\begin{aligned} 5x_1 - 6x_2 &\leq -3 \\ -x_1 - 6x_2 &\leq -2 \\ x_1, x_2 &\leq 1 \\ x_1, x_2 &\geq 0. \end{aligned}$$

a) Apply the method of lift and project to compute the polyhedron $P_1$. Sketch both $P$ and $P_1$.

b) Apply the method of lift and project to compute the integer hull $P_I$ of $P$.

**Exercise 4.3**
Let $a, c \in \mathbb{R}^n$, $b \in \mathbb{N}$ be a knapsack problem and let

$$\mathcal{K}(a,b) := \operatorname{conv}\left\{x \in \{0,1\}^n : a^T x \leq b\right\}$$

denote the *knapsack polytope*. A set $C \subset \{1, \ldots, n\}$ is called a *cover*, if $\sum_{j \in C} a_j > b$.

a) Show that the *knapsack cover inequalities*

$$\sum_{j \in C} x_j \leq |C| - 1 \tag{KNAP-C}$$

are valid inequalities for $\mathcal{K}(a,b)$.

b) Give an example that proves that the inequalities (KNAP-C) do not generally define a facet of $\mathcal{K}(a,b)$.

c) For a cover $C$, define

$$E(C) := C \cup \{j \in \{1, \ldots, n\} : a_j \geq a_i \ \forall i \in C\},$$

the *extended cover*. Derive a valid inequality for $\mathcal{K}(a,b)$ that involves $E(C)$. Give an example where your extended cover inequality is stronger than the cover inequality.

d) Give an example for an extended cover inequality that does not define a facet of $\mathcal{K}(a,b)$. Apply the lifting procedure discussed in the lecture until you obtain a facet.

**Please turn over!**

e) Can you find three examples with the following properties:

   1) The LP relaxation yields a non-integral optimal solution.

   2) For the first example, addition of some cover inequality leads to an integral optimal solution.

   3) For the second example, a cover inequality does not suffice, but an extended cover inequality does.

   4) For the third example, the extended cover inequality does not suffice, but its lifted version does.

f) Design an algorithm that separates cover inequalities, i. e., given a non-integral solution to the LP relaxation of the knapsack problem, your algorithm should find a violated cover inequality (or assert that there is none).

g) *Bonus exercise:* Use a programming language of your choice to implement a program that does the following:

   1) compute a solution for the LP relaxation of a knapsack problem

   2) as long as the the solution is not integral, separate some cover inequality, add it to the relaxation and resolve.

   As you will need an LP solver, Xpress Mosel or MatLab is recommended for this exercise.

h) *Bonus exercise:* Extend your program to not only add the separated cover inequality, but rather the lifted extended cover inequality. Does that make a difference in practice?

**Exercise 4.4**

Let $G = (V, E)$ be an undirected graph, $T \subset V$ a set of nodes (called "terminals") and $c : E \to \mathbb{N}$ a cost function on the edges of $G$. A *Steiner tree* in $G$ is a tree that spans all nodes in $T$. The STEINER TREE PROBLEM asks for a Steiner tree of minimum cost. Define the *Steiner tree polytope* by

$$P_S := \operatorname{conv}\{x \in \{0,1\}^m : x \text{ corresponds to a Steiner tree in } G\}.$$

a) Design an ILP model for the STEINER TREE PROBLEM. Use the binary variables $x_e, e \in E$, to indicate whether an edge is part of an optimal solution or not.

b) Let $C_1, \ldots, C_k$ be a partition of $V$ such that $T \cap C_i \neq \emptyset$ for all $i \in \{1, \ldots, k\}$. Show that the inequality

$$\sum_{e \in \delta(C_1, \ldots, C_k)} x_e \geq k - 1 \tag{ST}$$

   is valid for $P_S$. (Recall $\delta(C_1, \ldots, C_k)$ denotes the set of all edges with endpoints in two different $C_i$-sets, more precisely all edges $\{u, v\} \in E$ such that there are $i \neq j$ with $u \in C_i$ and $v \in C_j$.)

c) A graph is called 2-*connected* if at least two node-disjoint paths exist between any two nodes. Show the following auxiliary result: For a 2-connected graph $G = (V, E)$ with $|E| \geq 2$ and two edges $e, \tilde{e} \in E$ with $e \neq \tilde{e}$ there are spanning trees $T, \tilde{T}$ of $G$ such that $T \Delta \tilde{T} = \{e, \tilde{e}\}$.

d) Show that (ST) is facet defining for $P_S$ if the following two conditions hold:

   1) The subgraph induced by $C_i$ is connected for all $i \in \{1, \ldots, k\}$.

   2) The graph defined by shrinking each node set $C_i$ into a single node is 2-connected.

**This exercise sheet will be discussed in the tutorial classes on Dec 8 and Dec 15.**