Exercise Sheet 6

Exercise 6.1

Consider a knapsack problem with n items:

$$\max c^T x$$
$$a^T x \leq b,$$

where $a, c \in \mathbb{N}^n$ and $b \in \mathbb{N}$ and let $\varepsilon \in \mathbb{Q}_{>0}$ and $x^* \in \{0,1\}^n$ an optimal solution. In the following, a runtime bound for different approximation algorithms for the knapsack problem is given. Decide for each algorithm whether it is an ε-approximation algorithm, a PTAS and/or an FPTAS. Check *every* correct answer.

a) Algorithm $\mathcal{A}_\varepsilon$ computes a feasible solution x_0 such that $c^T x_0 \geq \frac{3}{4} c^T x^*$ in running time $2n^2$ for every $\varepsilon > 0$. Mark *every* correct answer:

- ☐ The algorithm is a 4-approximation.
- ☐ The algorithm is a $\frac{1}{4}$-approximation.
- ☐ The algorithm is a $\frac{3}{4}$-approximation.
- ☐ The algorithm is an ε-approximation.
- ☐ The algorithm is a PTAS.
- ☐ The algorithm is an FPTAS.
- ☐ None of the above answers is correct.

b) Algorithm $\mathcal{B}_\varepsilon$ computes a feasible solution x_0 such that $c^T x_0 \geq \frac{\varepsilon}{2} c^T x^*$ in running time $2n^2$ for every $2 > \varepsilon > 0$. Mark *every* correct answer:

- ☐ The algorithm is a $\frac{1}{2}$-approximation.
- ☐ The algorithm is an $\frac{\varepsilon}{2}$-approximation.
- ☐ The algorithm is an ε-approximation.
- ☐ The algorithm is a PTAS.
- ☐ The algorithm is an FPTAS.
- ☐ None of the above answers is correct.

c) Algorithm $\mathcal{C}_\varepsilon$ computes a feasible solution x_0 such that $c^T x_0 \geq \left(1 - \frac{\varepsilon}{2}\right) c^T x^*$ in running time bn for every $2 > \varepsilon > 0$. Mark *every* correct answer:

- ☐ The algorithm is a 2-approximation.
- ☐ The algorithm is a $\frac{1}{2}$-approximation.
- ☐ The algorithm is an $\frac{\varepsilon}{2}$-approximation.
- ☐ The algorithm is an ε-approximation.
- ☐ The algorithm is a PTAS.
- ☐ The algorithm is an FPTAS.
- ☐ None of the above answers is correct.

d) Algorithm $\mathcal{D}_\varepsilon$ computes a feasible solution x_0 such that $c^T x_0 \geq (1-\varepsilon) c^T x^*$ in running time $n^2 \varepsilon^n$ for every $\varepsilon > 0$. Mark *every* correct answer:

- ☐ The algorithm is an ε-approximation.
- ☐ The algorithm is a PTAS.
- ☐ The algorithm is an FPTAS.
- ☐ None of the above answers is correct.

e) Algorithm $\mathcal{E}_\varepsilon$ computes a feasible solution x_0 such that $c^T x_0 \geq (1-\varepsilon) c^T x^*$ in running time $n^2 \frac{1}{\varepsilon^2}$ for every $\varepsilon > 0$. Mark *every* correct answer:

- ☐ The algorithm is an ε-approximation.
- ☐ The algorithm is a PTAS.
- ☐ The algorithm is an FPTAS.
- ☐ None of the above answers is correct.

f) Algorithm $\mathcal{F}_\varepsilon$ computes a feasible solution x_0 such that $c^T x_0 \geq (1-\varepsilon) c^T x^*$ in running time $\max_i a_i \cdot n^2 \cdot \frac{1}{\varepsilon}$ for every $\varepsilon > 0$. Mark *every* correct answer:

- ☐ The algorithm is an ε-approximation.
- ☐ The algorithm is a PTAS.
- ☐ The algorithm is an FPTAS.
- ☐ None of the above answers is correct.

Exercise 6.2

Consider a knapsack problem

$$\begin{aligned} \max\ & c^T x \\ & a^T x \leq b \end{aligned}$$

with $a, c \in \mathbb{N}^n$ and $b \in \mathbb{N}$ such that $a \cdot \mathbb{1} \leq b$ and with the additional property that

$$\frac{c_1}{a_1} \geq \frac{c_2}{a_2} \geq \cdots \geq \frac{c_n}{a_n}.$$

Please see next page!

(Of course, for every knapsack problem the items can be re-ordered to guarantee this property, so this assumption can be made up to some polynomial computational effort.)

a) Consider the following algorithm for the above knapsack problem:

   1) Determine the maximum index $k \in \{1, \ldots, n\}$ such that $\{1, \ldots, k\}$ is a feasible knapsack.

   2) Return as a solution the knapsack $\{1, \ldots, k\}$.

   Which of the following is correct? Mark *every* correct answer (and only correct answers):

   - ☐ The algorithm determines an optimal solution to the knapsack problem.
   - ☐ The algorithm is a $\frac{1}{2}$-approximation for the knapsack problem.
   - ☐ The algorithm is a PTAS for the knapsack problem.
   - ☐ The algorithm is an FPTAS for the knapsack problem.
   - ☐ None of the above answers is correct.

b) Consider the following algorithm for the above knapsack problem:

   1) Determine the maximum index $k \in \{1, \ldots, n\}$ such that $\{1, \ldots, k\}$ is a feasible knapsack.

   2) Return as a solution either the knapsack $\{1, \ldots, k\}$ or the knapsack $\{k + 1\}$, depending on which choice gives the larger objective value.

   Which of the following is correct? Mark *every* correct answer (and only correct answers):

   - ☐ The algorithm determines an optimal solution to the knapsack problem.
   - ☐ The algorithm is a $\frac{1}{2}$-approximation for the knapsack problem.
   - ☐ The algorithm is a PTAS for the knapsack problem.
   - ☐ The algorithm is an FPTAS for the knapsack problem.
   - ☐ None of the above answers is correct.

Exercise 6.3

Which of the following statements is true? Mark *each* correct statement.

a)

   - ☐ Branch & bound for general integer linear programs is a PTAS.
   - ☐ A pure branch & bound algorithm always computes an optimal solution.
   - ☐ In a branch & bound algorithm, if some subproblem is infeasible then the original problem is also infeasible.
   - ☐ A cutting plane algorithm for ILP always terminates after a finite number of steps.
   - ☐ A cutting plane algorithm for ILP always terminates after a polynomially bounded (in the encoding size of the input) number of steps.
   - ☐ A cutting plane algorithm for ILP always converges to an optimal solution.
   - ☐ None of the above answers is correct.

Exercise 6.4

Which of the following statements is true? Mark *each* correct statement.

a)

- □ A facet of an n-dimensional polyhedron always contains n affinely independent points.
- □ A facet of an n-dimensional polyhedron always contains n affinely independent vertices.
- □ A facet of an n-dimensional polytope always contains n affinely independent integral points.
- □ A facet of an n-dimensional polytope always contains n affinely independent integral vertices.
- □ A facet of an n-dimensional integral polytope always contains n affinely independent integral vertices.
- □ None of the above answers is correct.

Exercise 6.5

a) Let $P \subset \mathbb{R}^n$ be a fully-dimensional integral polytope. Which of the following statements is true? Mark *each* correct statement. (Note: By "lifting" we refer to the process of lifting for each variable not yet contained in an inequality.)

- □ If an inequality $a^T x \leq b$ is valid for P, then lifting it yields a facet-defining inequality for P.
- □ If $P \subset [0,1]^n$ and an inequality $a^T x \leq b$ is valid for P, then lifting it yields a facet-defining inequality for P.
- □ If $P \subset [0,1]^n$ and the inequality $x_1 \leq 1$ contains a vertex of P, then lifting the inequality yields a facet-defining inequality for P.
- □ None of the above answers is correct.

You can check your answers for correctness online at `www-m9.ma.tum.de/WS2010/CombOpt` (use the link next to the sheet download). Please send questions and feedback on this exercise sheet to `m.ritter@ma.tum.de`. We will discuss the topics of this exercise sheet and do some repetition in the exercise classes on Jan 19 and Jan 26, 2011. For exam practice, there will be another multiple choice exercise sheet at the end of the semester.