Exercise Sheet 9

Exercise 9.1

Consider a TRAVELING SALESMAN PROBLEM (TSP) on a complete graph $K_n = (V, E)$ on $n \in \mathbb{N}$ nodes with edge lengths $c : E \to \mathbb{N}$. An ILP description of the problem is

$$\min \sum_{e \in E} c_e x_e$$

$$\sum_{e \in \delta(\{v\})} x_e = 2 \quad \text{for all } v \in V \tag{1}$$

$$\sum_{e \in E(S)} x_e \leq |S| - 1 \quad \text{for all } S \subset V, \emptyset \neq S \neq V \tag{2}$$

$$x \in \{0,1\}^m .$$

Which of the following statements are true, which are false? Try to find a reasoning for your answer! (Recall: For a subgraph T of K_n and some node $i \in V$, the set of all edges in T incident to i is denoted by $\delta_T(i)$.)

	true	false
a) For every $\lambda \in \mathbb{R}^n$, the following value is a lower bound for the length of an optimal tour: $$\min \left\{ \sum_{e \in T} c_e + \sum_{i=2}^{n} (\lvert\delta_T(i)\rvert - 2) \cdot \lambda_i : T \text{ is a 1-tree} \right\}$$	☐	☐
b) The Lagrangean dual of the above ILP formulation obtained through relaxation of the constraints (1) provides an upper bound for the length of an optimal TSP tour.	☐	☐
c) The Lagrangean dual of the above ILP formulation obtained through relaxation of the constraints (1) provides a lower bound for the length of an optimal TSP tour.	☐	☐
d) The Lagrangean dual of the above ILP formulation obtained through relaxation of the integrality constraints provides an upper bound for the length of an optimal TSP tour.	☐	☐
e) The Lagrangean dual of the above ILP formulation obtained through relaxation of the integrality constraints provides a lower bound for the length of an optimal TSP tour.	☐	☐

Exercise 9.2

Which of the following statements on the method of "lift and project" are true, which are false? In all statements, let P denote some initial polytope contained in $[0,1]^n$ and let P_j denote the polytope obtained from P through a lift and project step on the j-th coordinate.

	true	**false**
a) The polytope P_j has the property $P_j \subset P$.	☐	☐
b) The polytope P_j has the property $P_j \subsetneq P$.	☐	☐
c) The number of facet of P_j is linear in the number of facets of P.	☐	☐
d) The number of facet of P_j is polynomial in the number of facets of P.	☐	☐
e) $\dim P_j \leq \dim P$	☐	☐
f) $\dim P_j = \dim P$	☐	☐
g) If P is an integral polytope, then so is P_j.	☐	☐
h) If P has only one non-integral vertex, then P_j is an integral polytope.	☐	☐
i) The method of lift and project yields an integral polytope after at most n lift and project steps.	☐	☐

Exercise 9.3

Let $G = (V, A)$ be a directed graph on the node set $V = \{1, \ldots, n\}$, $c : A \to \mathbb{N}$ a cost function on the arcs of G.

Please see next page!

a) Consider the following ILP, where both the x- and y-variables are variables on the arcs of G:

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in A} c_{ij} y_{ij} \\
& \sum_{j=1}^{n} y_{ij} = 1 \quad \forall i = 1, \dots, n \\
& \sum_{i=1}^{n} y_{ij} = 1 \quad \forall j = 1, \dots, n \\
& \sum_{j=1}^{n} x_{1j} = n - 1 \\
& \sum_{i=1}^{n} x_{ij} = \sum_{i=1}^{n} x_{ji} + 1 \quad \forall j = 2, \dots, n \\
& x_{ij} \le (n-1) y_{ij} \\
& x \ge 0 \\
& y \in \{0,1\}^{|A|},
\end{aligned}
$$

Which problem is encoded by this model?

- ☐ Find a minimum cost flow of value 1 from node 1 to node n!
- ☐ Find a minimum cost matching in the graph G (neglecting the direction of the arcs)!
- ☐ Find a minimum cost matching in the graph G such that all matching arcs (i, j) have the property $i < j$!
- ☐ Find a minimum cost TSP tour in G!
- ☐ Find a minimum cost covering (i e, containing all nodes) of G by disjoint directed cycles!
- ☐ Find a selection of arcs in G of minimum cost such that removal of these arcs disconnects G!
- ☐ Find a partition of G into two disjoint node sets such that the arcs between both sets have minimum cost (i e, a minimum cut of G)!

b) Consider the following ILP, where the x-variables are variables on the arcs of G and the u-variables are variables on the nodes of G:

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \forall i = 1,\dots,n$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \forall j = 1,\dots,n$$

$$u_i - u_j + nx_{ij} \leq n-1 \quad \forall i,j \in \{1,\dots,n\} \text{ with } i \neq j$$

$$x \in \{0,1\}^{|A|}$$

$$u \in \mathbb{R}^n$$

Which problem is encoded by this model?

- ☐ Find a minimum cost flow of value 1 from node 1 to node n!
- ☐ Find a minimum cost matching in the graph G (neglecting the direction of the arcs)!
- ☐ Find a minimum cost matching in the graph G such that all matching arcs (i,j) have the property $i < j$!
- ☐ Find a minimum cost TSP tour in G!
- ☐ Find a minimum cost covering (i e, containing all nodes) of G by disjoint directed cycles!
- ☐ Find a selection of arcs in G of minimum cost such that removal of these arcs disconnects G!
- ☐ Find a partition of G into two disjoint node sets such that the arcs between both sets have minimum cost (i e, a minimum cut of G)!

The exercises on this sheet will not be discussed in tutorial classes. Instead, you can check your answers online, see the link on the homepage at `www-m9.ma.tum.de/WS2010/CombOpt`.