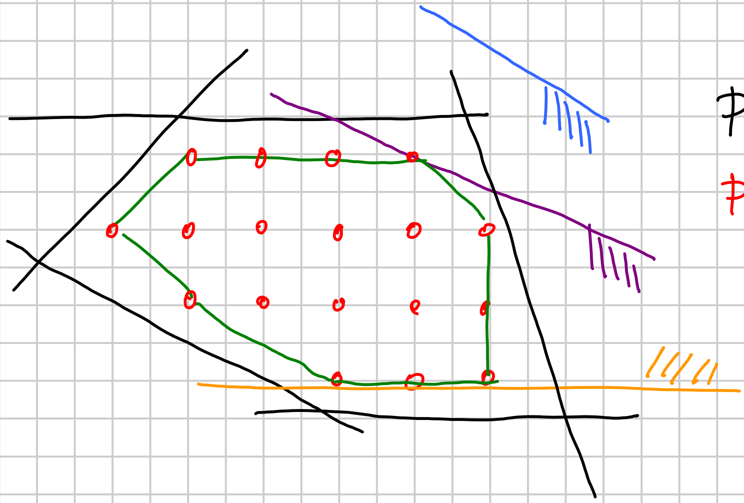
(2.) Method to strengthen ILP-formulations.

Lec 5, 22.11.12

(2.2) Generating valid inequalities ✓

now: **(2.3) Generating facet defining inequalities**

idea:



P

$P_I := P \cap \mathbb{Z}^n$

$\mathrm{conv}(P_I)$

find a description of $\mathrm{conv}(P_I)$
by valid inequalities
face-defining inequalities
→ facet-defining inequalities
(better approximation of $\mathrm{conv}(P_I)$).

in other words:

if $k = \dim(P_I) = \max\{\ell : \exists\, \ell+1$ affinely indep. points in $P_I\}$,

then find an inequality $a^T x \le \alpha$ which is
- valid for P_I and
- has k affinely indep. points in P_I that satisfy $a^T x = \alpha$.

Example 2.8 $G = (V, E)$ graph, $|V| = n$,

weights w_i for all $i \in V$.

recall: $U \subset V$ is a stable set iff $\forall v \neq w \in U: \{v,w\} \notin E$
clique $\in E$

maximal clique $:\Leftrightarrow$ U clique and
$\forall v \in V \setminus U: U \cup \{v\}$ not a clique

maximum clique $:\Leftrightarrow$ U clique and $\nexists$ clique U' in G with $|U'| > |U|$.

maximum weight clique $:\Leftrightarrow$ U clique and $\sum_{i \in U} w_i$ maximum value among all cliques.

task: find a maximum weight stable set in G.

model: $P := \{ x \in \mathbb{R}^n : \quad 0 \leq x_i \leq 1 \quad \forall i \in V$
$x_i + x_j \leq 1 \quad \forall \{i,j\} \in E \}$

is called the stable set polytope of G

$P_I = \{ \quad .... \quad x_i \in \{0,1\} \quad .... \quad \}$

find: $\max \{ \sum_{i \in V} w_i x_i : x \in P_I \}$.

Observation 2.9 If U is a clique in G, then $\sum_{i \in U} x_i \leq 1$ is a valid inequality for P_I.

(but not necc. for P! e.g. $x_i := 1/2 \quad \forall i \in V$)

Proof: $\forall x \in P_I$ $\forall$ clique U there can at most one $i \in U$ with $x_i = 1$ because otherwise $x_i + x_j = 2$ but $\{i,j\} \in E$. $\square$

Proposition 2.10 Let U be a clique in G. Then

$\sum_{i \in U} x_i \leq 1$ defines a facet in P_I

$\iff$ U is a maximal clique in G.

Proof: $\vec{0} \in P_I$, for $e^i := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ i-th component $\in P_I$ $\forall i \in V$

thus $\dim(P_I) = n$.

Wlog $U = [k] = \{1, \dots, k\}$
$e^1, \dots, e^k$ satisfy $\sum_{i \in U} x_i = 1$.

($\Leftarrow$) U is a maximal clique
$\Rightarrow \forall j \in V \setminus U \ \exists\, r(j) \in U : \{j, r(j)\} \notin E$

Then $\forall \ell \in V \setminus U$ let $x^\ell := \begin{array}{c} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \end{array} \begin{array}{l} \\ \leftarrow r(\ell) \\ \\ \leftarrow \ell \\ \\ \end{array}$,

(because $x_i + x_j \leq 1$)

thus $x^\ell \in P_I$ and x^ℓ satisfies $\sum_{i \in U} x_i = 1$ for all $\ell \in V \setminus U$.

$\Rightarrow$ $e^1, \dots, e^k, x^{k+1}, \dots, x^n$ all satisfy $\sum_{i \in U} x_i = 1$

and are lin. indep., hence aff. indep.,

$\Rightarrow$ $\sum_{i \in U} x_i \leq 1$ is a facet-defining inequality.

$\Rightarrow$ Assume that U is not maximal.

$\Rightarrow$ $\exists \ell \in V \setminus U$ s.t. $U \cup \{\ell\}$ is a clique

$\overset{2.9}{\Rightarrow}$ $\sum_{i \in U \cup \{\ell\}} x_i \leq 1$ is valid for P_I

but summing

$$-x_\ell \leq 0 \qquad (\text{valid for } P_I)$$

$$+ \quad \sum_{i \in U \cup \{\ell\}} x_i \leq 1 \qquad (\text{valid for } P_I)$$

$$\overline{\qquad \sum_{i \in U} x_i \leq 1 \qquad}$$

gives $\sum_{i \in U} x_i \leq 1$,

so this cannot be a facet-defining inequality.

Observation 2.11

If $C \subset V$ is an odd cycle in G, then

$$\sum_{i \in C} x_i \le \frac{|C|-1}{2}$$ is a valid inequ. for P_I.

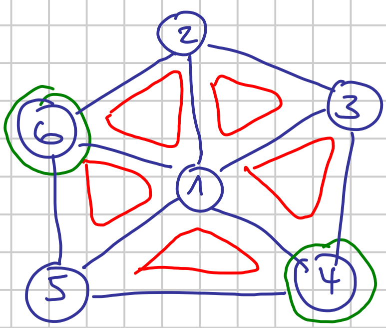
Proof: $\alpha(G[C]) \le \frac{|C|-1}{2}$.

So if there were more than $\frac{|C|-1}{2}$ components in x equal to 1, then they could not form a stable set and hence there would be an edge $\{i,j\}$ with $x_i + x_j = 2$, $\lightning$. $\square$

Now we introduce LIFTING, which derives a higher-dim. face from a lower-dim. face.

Example 2.12

$G =$



with unit weights.
What is a maximum weight stable set in G?

$P =$ stable set polytope of $G \subset \mathbb{R}^6$
$\dim(P) = 6$

What are the inequ. induced by maximal cliques in G?

$$
\begin{array}{l}
x_1 + x_2 + x_3 \ldots \ldots \le 1 \\
x_1 + x_3 + x_4 \le 1 \\
x_1 + x_4 + x_5 \le 1 \\
x_1 + x_5 + x_6 \le 1 \\
x_1 + x_2 + \qquad + x_6 \le 1
\end{array}
$$

add these inequ. here

One can check that

$$\max\left\{ \sum_{i=1}^{6} x_i : 0 \le x_i \le 1, \text{ such that (and } x_i + x_j \le 1 \quad \forall \{i,j\} \in E) \right\}$$

has a unique opt. soln. at $x^0 := \frac{1}{2}(0, 1, 1, 1, 1, 1) \notin P_I$.

$\Rightarrow$ max. clique inequ. not enough to describe $\mathrm{conv}(P_I)$.

Consider the odd cycle $C := \{2, 3, 4, 5, 6\}$ with the inequality

$$x_2 + x_3 + x_4 + x_5 + x_6 \le \frac{5-1}{2}, \qquad (*)$$

which is not satisfied by x^0!

$(*)$ is satisfied with equality by the max. clique vectors

$$
\begin{array}{ll}
x^{24} := & (0\ 1\ 0\ 1\ 0\ 0) \\
x^{25} := & (0\ 1\ 0\ 0\ 1\ 0) \\
x^{35} := & (0\ 0\ 1\ 0\ 1\ 0) \\
x^{36} := & (0\ 0\ 1\ 0\ 0\ 1) \\
x^{46} := & (0\ 0\ 0\ 1\ 0\ 1)
\end{array}
$$

which are lin. independent,
but they only give us a 4-dim face,
because we cannot find any other point in P_I
that satisfies $(*)$ with equality.

Need a better inequality?

Note: $(*)$ is facet-defining for

$\mathrm{conv}\left(P_I \cap \{ x \in \{0,1\}^6 : x_1 = 0 \} \right)$

which has dim = 5.

idea: lift $(*)$ so that it becomes
facet-defining for $\mathrm{conv}(P_I)$.