grade:

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seat number

I hereby confirm that the exam sheet I have received is complete. I have checked the sheet and have not noticed any obvious printing errors or missing pages.

student's signature

Technische Universität München
Zentrum Mathematik

Combinatorial Optimization (MA 4502)
WS 2012/13

Prof. Dr. Anusch Taraz
Feb 25, 2013

Please read the following instructions carefully:

- Please check your exam sheets: The exam consists of **10 printed pages** (including cover sheet and summary sheets) with **5 problems**.
- You have 60 minutes to complete the exam. An announcement will be made 15 minutes before examination time ends.
- Please answer each problem in the space immediately following the problem statement. If you need more space, be sure to clearly mark where your answer continues.
- Give precise reasoning for all of your answers (unless explicitly stated otherwise). Answers will be accepted in both English and German.
- Presumably 17 out of 40 credits will be necessary to pass the exam.
- This is a closed-book examination—no utilitites or devices beyond writing utensils are allowed during the exam. Failure to comply with these rules will result in a grade of 5.0 ("fail").

for supervisory staff only:

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I	II

Problem 1 [≈ 12 credits]
Consider the knapsack problem

$$\begin{aligned} \max\ & x_1 + 2x_2 + 5x_3 + 2x_4 \\ & x_1 + 2x_2 + 4x_3 + 2x_4 \leq 7 \\ & x \in \{0,1\}^4 \end{aligned}$$

Use a dynamic programming algorithm (either from the lecture or from the tutorial classes) to compute an optimal solution for this problem.

a) Give a brief explanation of your notation and write down the recursion you will use.

b) State the initial values that you use to start your recursion.

c) Perform the computations. Write down enough information to make each step of your computations comprehensible!

d) What is the optimal solution? State both an optimal knapsack and its value.

I	II

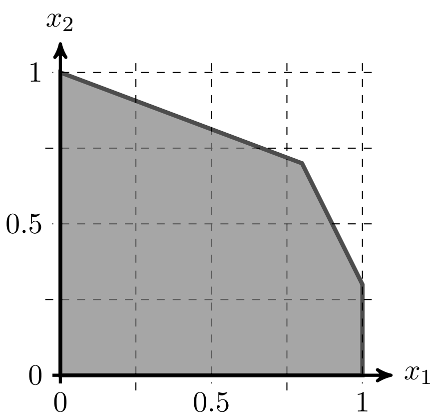
Problem 2 [≈ 6 credits]
Consider the polyhedron $P := \{x \in \mathbb{R}^2 : Ax \leq b\} \subset \mathbb{R}^2$ with

$$A := \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ -1 & 2 \\ -1 & -2 \\ 2 & 1 \end{pmatrix}, \quad b := \begin{pmatrix} 3 \\ -1 \\ 4 \\ -3 \\ 8 \end{pmatrix}.$$

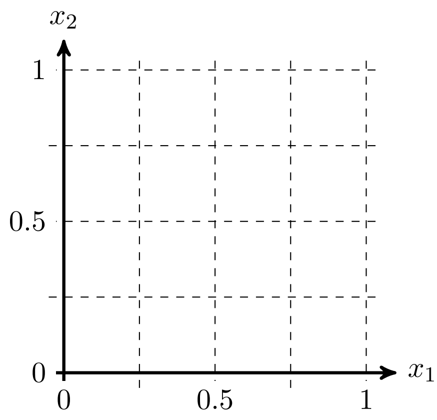
Use the method of rounding to devise a valid inequality for the integer hull $\operatorname{conv}(P_I)$ of P that cuts off the point $x^* = (1, \frac{5}{2}) \in P$.

I	II

Problem 3 [≈ 4 credits]
Consider the polytope P depicted below. Let P' be the polytope resulting from one lift-and-project step applied to P with respect to the variable x_1. Sketch P' into the empty coordinate system. *You do not need to give any reasoning for this problem!*



P



P'

I	II

Problem 4 [≈ 1+2+4+3+2 credits]
Let $G = (V, E)$ be a connected graph on $n = |V|$ nodes. We denote by $\alpha(G)$ the stability number of G, i. e. the size of a largest stable set in the graph G. Consider the stable set polytope

$$P_{\mathrm{STAB}}(G) := \operatorname{conv}\left\{x \in \{0,1\}^n : x_u + x_v \leq 1 \text{ for all edges } \{u,v\} \in E\right\}.$$

a) Prove that the inequality

$$\sum_{v \in V} x_v \leq \alpha(G)$$

is valid for P_{STAB}.

b) Show that $\dim(P_{\mathrm{STAB}}) = n$.

continued on next page ...

Problem 4 continued

An edge $e \in E$ is called a *critical edge*, if the stability number of G increases after deletion of e, i. e. if $\alpha((V, E \setminus \{e\})) > \alpha(G)$.

c) Let $e' = \{u, v\} \in E$ be a critical edge of G. Show that there are stable sets $S_u, S_v \subset V$ in G such that the following holds:

   i) $|S_u| = |S_v| = \alpha(G)$

   ii) $u \in S_u$ and $v \in S_v$

   iii) $S_u \setminus \{u\} = S_v \setminus \{v\}$

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Problem 4 continued

Let $E' \subset E$ be the subset of all critical edges of the graph G and let

$$F := \left\{ x \in \mathcal{P}_{\text{STAB}} : \sum_{v \in V} x_v = \alpha(G) \right\}.$$

d) Use c) to prove the following statement: If E' is connected and if there are $c \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$ such that $c^T x = \gamma$ holds for all $x \in F$, then there is some constant $\lambda \in \mathbb{R}$ such that $c = \lambda \mathbb{1}$ and $\gamma = \lambda \alpha(G)$.

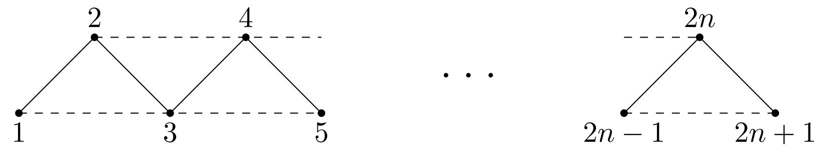
e) Use d) to show: If E' is connected, the inequality

$$\sum_{v \in V} x_v \leq \alpha(G)$$

induces a facet of P_{STAB}.

I	II

Problem 5 [≈ 6 credits]
In the lecture we proved that the Christofides heuristic for TSP is a $\frac{3}{2}$-approximation on complete metric graphs. Show that this bound is asymptotically tight. Use the graph depicted below, define suitable edge weights on the solid and dashed edges and then consider the metric completion of your graph to prove the statement.



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