Problem 1:
Let $n \in \mathbb{N}$. We want to construct a polytope $P = \left\{x \in \mathbb{R}^2 \mid Ax \leq b\right\} \subset \mathbb{R}^2$ such that the Chvátal-rank of P is n.

a) Let $A_1 = \begin{pmatrix} 0 & -1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix}$ and $b_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Show that $P_1 = \left\{x \in \mathbb{R}^2 \mid A_1 x \leq b_1\right\}$ has Chvátal-rank 1, i.e. $P_1' = (P_1)_I$.

b) Let $A_n = \begin{pmatrix} 0 & -1 \\ -n & 1 \\ n & 1 \end{pmatrix}$ and $b_n = \begin{pmatrix} 0 \\ 0 \\ n \end{pmatrix}$ and $P_n = \left\{x \in \mathbb{R}^2 \mid A_n x \leq b_n\right\}$.

Show that $P_n' = P_{n-1}$. *(Hence by induction P_n has Chvátal-rank n.)*

Hint (from Discrete Optimization): Instead of adding a new cut $\sum_{j \in [n]} \lfloor u^T(a^j) \rfloor x_j \leq \lfloor u^T b \rfloor$ *for every* $u \in \mathbb{R}^n_{\geq 0}$ *it suffices to add cuts* $h^T x \leq h^T \hat{x}$*, where* $\hat{x}$ *is a fractional vertex of* P *and* h *any element of a Hilbert-basis* H *of the normal cone of* $\hat{x}$*.* $H = \{h^1, \ldots, h^l\}$ *is a Hilbert-basis of a cone* C*, if every* $c \in C \cap \mathbb{Z}^n$ *can be represented as* $c = \sum_{i \in l} \lambda_i h^i$ *where* $\lambda_i \in \mathbb{Z}_{\geq 0}$*,* $i = 1, \ldots, l$*.*

Solution:

a) We compute the vertices of P_1:

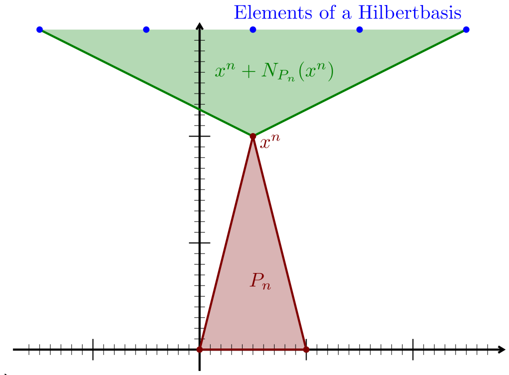
I		Vertex
$\{1,2\}$	$\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} \bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\{1,3\}$	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \tilde{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\tilde{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\{2,3\}$	$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} x^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$x^{(1)} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$

The vertex $x^{(1)}$ is fractional, the Hilbert basis H_1 of its normal cone is:
$H_1 = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$.
We get a new cut for every $h \in H_n$: $h^T x \leq \lfloor h^T x^{(1)} \rfloor$, thus:

$$P_1' = \left\{ x \in \mathbb{R}^2 \mid \begin{pmatrix} 0 & -1 \\ -1 & 1 \\ 1 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} = \operatorname{conv} \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} = (P_1)_I$$

b) Again, we compute the vertices of P_n:

I		Vertex
$\{1,2\}$	$\begin{pmatrix} 0 & -1 \\ -n & 1 \end{pmatrix} \bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\{1,3\}$	$\begin{pmatrix} 0 & -1 \\ n & 1 \end{pmatrix} \tilde{x} = \begin{pmatrix} 0 \\ n \end{pmatrix}$	$\tilde{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\{2,3\}$	$\begin{pmatrix} -n & 1 \\ n & 1 \end{pmatrix} x^{(n)} = \begin{pmatrix} 0 \\ n \end{pmatrix}$	$x^{(n)} = \begin{pmatrix} 0.5 \\ 0.5n \end{pmatrix}$



The normal cone of $x^{(n)}$ is: $N_{P_n} = \text{pos}\left\{ \begin{pmatrix} -n \\ 1 \end{pmatrix}, \begin{pmatrix} n \\ 1 \end{pmatrix} \right\}$.

The Hilbert basis H_n of $N_{P_n}(x^{(n)})$ is $H_n = \left\{ h_i = \begin{pmatrix} i \\ 1 \end{pmatrix} \mid i = 0, \pm 1, \dots, \pm n \right\}$.

For $h_i \in H_n$ we get the new inequality: $h_i^T x \leq \lfloor h_i^T x^{(n)} \rfloor = \left\lfloor \begin{pmatrix} i \\ 1 \end{pmatrix}^T \begin{pmatrix} 0.5 \\ 0.5n \end{pmatrix} \right\rfloor = \lfloor \frac{i+n}{2} \rfloor$.

In particular for $i = -(n-1)$: $\begin{pmatrix} -n+1 \\ 1 \end{pmatrix}^T x \leq \lfloor \frac{-n+1+n}{2} \rfloor = 0$

and for $i = n-1$: $\begin{pmatrix} n-1 \\ 1 \end{pmatrix}^T x \leq \lfloor \frac{n-1+n}{2} \rfloor = n-1$.

Hence we have $P'_n \subset P_{n-1} = \left\{ \begin{pmatrix} 0 & -1 \\ -n+1 & 1 \\ n-1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 0 \\ 0 \\ n-1 \end{pmatrix} \right\}$ and "$\supset$", if we can show that all other inequaltities $h_i^T x \leq \lfloor \frac{i+n}{2} \rfloor$ are redundant for P_{n-1}. Indeed, for $i \in \pm[n]$ it holds that:

$h_i^T \bar{x} = \begin{pmatrix} i \\ 1 \end{pmatrix}^T \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \leq \lfloor \frac{i+n}{2} \rfloor$,

$h_i^T \tilde{x} = \begin{pmatrix} i \\ 1 \end{pmatrix}^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = i \leq \lfloor \frac{i+n}{2} \rfloor$ and

$h_i^T x^{(n-1)} = \begin{pmatrix} i \\ 1 \end{pmatrix}^T \begin{pmatrix} 0.5 \\ 0.5(n-1) \end{pmatrix} = \frac{i+n-1}{2} \leq \lfloor \frac{i+n}{2} \rfloor$. Thus:

$P_{n-1} = \text{conv}\{\bar{x}, \tilde{x}, x^{(n-1)}\} \subset \left\{ x \in \mathbb{R}^2 \mid \begin{pmatrix} i \\ 1 \end{pmatrix}^T x \leq \lfloor \frac{i+n}{2} \rfloor, i \in \pm[n] \text{ and } \begin{pmatrix} -1 \\ 0 \end{pmatrix}^T x \leq 0 \right\} = P'_n$.

Problem 2:

The Wandering Salesperson Problem (WSP) is given by:

input: $n \in \mathbb{N}$,
complete graph: $G = (V, E)$, $V = [n]$, $E = \binom{[n]}{2}$
cost function: c: $E \to \mathbb{R}_{\geq 0}$

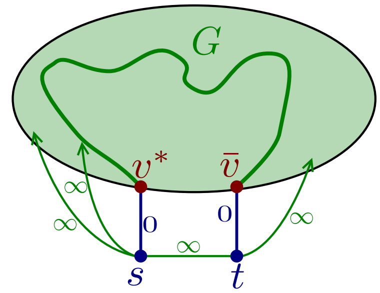
task: compute a start s and a destination vertex $t \neq s$ and a s-t-path that visits every vertex $v \in V$ exactly once such that (s, t, P) is minimal over all possible choices.

Show that (WSP) is $\mathbb{NP}$-hard.

Solution: We show (TSP) $\leq_p$ (WSP): Let $I = (n, V, E, c)$ an instance of (TSP) and let $v^* \in V$ an arbitrary vertex. We now add two dummy-vertices s and t and define the costs of their adjacent edges in a way that forces the algorithm to use these two as start/terminal point:

Define $G' = (V', E')$ with $V' = V \cup \{s, t\}$ and $E' = \binom{V'}{2}$
and define for every $\bar{v} \in V$:

$c'_{\bar{v}}(s, v) = \infty \quad \forall v \in V \cup \{t\} \setminus \{v^*\}$
$c'_{\bar{v}}(t, v) = \infty \quad \forall v \in V \setminus \{\bar{v}\}$
$c'_{\bar{v}}(s, v^*) = 0$
$c'_{\bar{v}}(t, \bar{v}) = 0$
$c'_{\bar{v}}(v, w) = c(v, w) \quad \forall v, w \in V$



Any WSP-path in $(G', c'_{\bar{v}})$ that doesn't use s and t as starting/terminal point has costs ∞. Let $P_{\bar{v}}$ the optimal WSP-path in $(G', c'_{\bar{v}})$. We can transform it to a Hamilton-cycle $T_{\bar{v}}$ in G by removing the vertices s and t and adding the edge $(v^*, \bar{v})$. (Recognize that v^* is visited right after s, since this is the only edge adjacent to s with costs $< \infty$, in fact it has costs 0 and this way we forced the algorithm to use v^* as starting/terminal point in G. Analogous for t and $\bar{v}$). $T_{\bar{v}}$ has costs $c(T_{\bar{v}}) = c'_{\bar{v}}(P_{\bar{v}}) + c(v^*, \bar{v})$. The tour $T^* = \underset{\bar{v} \in V \setminus \{v^*\}}{\operatorname{argmin}}\, c(T_{\bar{v}})$ is a minimal TSP-tour: Let S be a minimal TSP-tour and $(v^*, v') \in E$ one of the edges that S uses to leave v^*. Then $S \setminus (v^*, v')$ is a v^*-v-path that visits all other vertices and the following holds: $c(S) = c(S \setminus (v^*, v')) + c(v^*, v') \geq c(P_{v'}) + c(v^*, v') = c(T_{v'}) \geq \min_{\bar{v} \in V \setminus \{v^*\}} c(T_{\bar{v}}) = c(T^*)$.

Note that constructing T^* is done polynomially in n and only requires $n - 1$ times the solution of WSP on a graph G' that is only a polynomial factor larger than G ($|V'| = |V| + 2$, $|E'| = |E| + 2n + 1$). Thus $\min_{\bar{v} \in V \setminus \{v^*\}} c(T_{\bar{v}})$ can be computed in $O(n * t(n + 2))$, where $t(n)$ is the time needed to solve WSP on a graph with n vertices.