Combinatorial Optimization (MA 4502)

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exercise sheet 3

Exercise 3.1 (FPAS for the Knapsack Problem)
Let $v = (22, 64, 48, 100)^T$, $w = (2, 4, 3, 4)^T$ and $b = 6$ define a 0-1-knapsack problem. Apply the FPAS from the lecture to the problem for $\varepsilon = 1/2$.

Exercise 3.2 (Odd Set Inequalities for Maximum Matching)
Let $G = (V, E)$ be a graph with n vertices and m edges. The MAXIMUM MATCHING PROBLEM asks for a solution of the integer linear program

$$\begin{aligned} \max\ & \mathbb{1}^T x \\ & \sum_{e \in \delta(\{v\})} x_e \leq 1 \quad \text{for all } v \in V \\ & x \in \{0,1\}^m , \end{aligned}$$

where $\delta(S) := \{e \in E : |e \cap S| = |e \cap (V \setminus S)| = 1\}$ for any subset $S \subset V$ of vertices. The integral hull of this problem is called *matching polytope*.

Apply the method of rounding to the LP relaxation of the problem above to derive a valid inequality for the matching polytope that is a cutting plane for the LP relaxation. Can you give a combinatorial explanation of your inequality?

Exercise 3.3 (An Example of Odd Set Inequalities)
Let $G = (V, E)$ be the graph depicted in Fig. 1. A *maximum matching* is an optimal solution of the ILP

$$\begin{aligned} \max\ & \mathbb{1}^T x \\ & \sum_{e \in \delta(v)} x_e \leq 1 \quad \text{for all } v \in V \\ & x \in \{0,1\}^n , \end{aligned}$$

and the *matching polytope* $\mathcal{M}(G)$ is the convex hull of the feasible set of this problem.

a) Use a straightforward branch and bound algorithm for ILP to compute a maximum matching in G. (In case you are in need of an LP solver, a very simple online solver can be found at `http://www.zweigmedia.com/RealWorld/simplex.html`. You can of course also use Xpress/Mosel, a free student version can be found at `http://optimization.fico.com/student-version-of-fico-xpress.html`, or Matlab.)

b) Add the odd set inequality for V (see Exercise 3.2) to the LP relaxation. Does this help to speed up your branch and bound algorithm?

Please turn over.

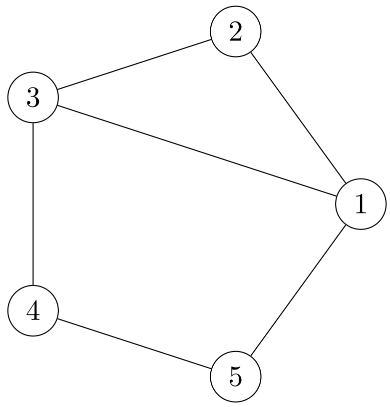

Figure 1: graph for Exercise 3.3

Exercise 3.4 (Comb Inequalities for TSP)
Let $G = (V, E)$ be a graph on n vertices and m edges and $c : E \to \mathbb{N}$ a cost function on the edges. The TRAVELING SALESMAN PROBLEM asks for a solution of the integer linear program

$$\min \sum_{e \in E} c_e x_e$$

$$\sum_{e \in \delta(\{v\})} x_e = 2 \quad \text{for all } v \in V \tag{1}$$

$$\sum_{e \in E(S)} x_e \leq |S| - 1 \quad \text{for all } S \subset V,\, \emptyset \neq S \neq V \tag{2}$$

$$x \in \{0,1\}^m .$$

The convex hull of all integer solutions P_{TSP} is called the *traveling salesman polytope.* A *comb* is a subgraph of G generated by node sets H ("handle") and $T_1, \ldots, T_t$ ("teeth") (with t odd and $t \geq 3$) such that the following holds:

i) the sets $T_1, \ldots, T_t$ are pairwise disjoint

ii) $2 \leq |T_i| \leq n - 2$ for all $i \in \{1, \ldots, t\}$

iii) for each $i \in \{1, \ldots, t\}$: $|H \cap T_i| \geq 1$ and $|T_i \setminus H| \geq 1$

a) Use the degree constraints (1) to prove that the inequality

$$2 \sum_{e \in E(H)} x_e + \sum_{i=1}^{t} \sum_{e \in \delta(H) \cap E(T_i)} x_e \leq 2\,|H| \tag{3}$$

is valid for P_{TSP}.

b) Use (3) and the subtour elimination constraints (2) to derive the valid inequality

$$\sum_{e \in E(H)} x_e + \sum_{i=1}^{t} \sum_{e \in E(T_i)} x_e \leq |H| + \frac{1}{2} \sum_{i=1}^{t} [(|T_i| - 1) + (|H \cap T_i| - 1) + (|T_i \setminus H| - 1)] . \tag{4}$$

c) Using (4), show that the *comb inequality*

$$\sum_{e \in E(H)} x_e + \sum_{i=1}^{t} \sum_{e \in E(T_i)} x_e \leq |H| + \sum_{i=1}^{t} (|T_i| - 1) - \frac{t+1}{2}$$

is valid for the traveling salesman polytope. Give a combinatorial interpretation of that inequality.

This exercise sheet will be discussed in the tutorials on Nov 21 and Nov 22.