Combinatorial Optimization (MA 4502)

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exercise sheet 6

Exercise 6.1 (A Modified CONTRACT Algorithm)
The CONTRACT algorithm discussed in the lecture has a success probability of $\Omega(\frac{1}{n^2})$. Suppose we modify the algorithm in such a way that it chooses a pair of *vertices* uniformly at random and contracts those (instead of a random edge). Show that there are input graphs where this modified algorithm has exponentially small success probability.

Exercise 6.2 (Running Time of FAST-CUT)
Let $T : \mathbb{N} \to \mathbb{R}$ be a monotonically increasing function that satisfies the recursion

$$T(n) \le 2dn^2 + 2T\left(\left\lceil 1 + \frac{n}{\sqrt{2}} \right\rceil\right)$$

for all $n \in \mathbb{N}$. Show that with $A := 20d \cdot T(2^{10})$ the following inequality holds:

$$T(n) \le An^2 \log_2(n)$$

Exercise 6.3 (Success Probability of FAST-CUT)

a) Suppose that $H = (V, E)$ is a graph on t vertices to which we apply the CONTRACT algorithm until we get to a graph H' with exactly

$$t' := \left\lceil 1 + \frac{t}{\sqrt{2}} \right\rceil$$

vertices. Let $C \subset E$ be an arbitrary mincut of H. Use Lemma 4.7 from the lecture to prove that the probability that C survives the contractions leading from H to H' is at least $\frac{1}{2}$.

b) For a graph H, let $P(H)$ denote the probability that FAST-CUT finds a mincut in H. Let H_1 and H_2 be the intermediate graphs computed by FAST-CUT in line 5) of Algorithm 4.8 from the lecture. Show that

$$P(H) \ge 1 - \left(1 - \frac{1}{2}P(H_1)\right) \cdot \left(1 - \frac{1}{2}P(H_2)\right).$$

c) Consider the recursive tree for FAST-CUT whose vertices correspond to intermediate graphs. The height of a graph in this tree is defined as its shortest distance to a leaf. Let $p(h)$ be the probability that FAST-CUT finds a mincut in a graph of height h. Use induction on h to prove that

$$p(h) \ge \frac{1}{h+1}.$$

d) Combine the previous results to deduce that the success probability of FAST-CUT is $\Omega\left(\frac{1}{\log(n)}\right)$.

This exercise sheet will be discussed in the tutorials on Jan 16 and Jan 17.