Technische Universität München
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Discrete Optimization (MA 3502), WiSe 2012/13

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Problem Sheet 1

Problem 1.1

Let $G = (V, E)$ a graph with node set $V = [n]$ and $n \geq 2$. Consider the following LP:

$$\begin{array}{rrcll} \max & \sum_{i\in[n]} x_i & & & \\ & x_i + x_j & \leq & 1 & \text{for all } \{i,j\} \in E \ , \qquad (P_G) \\ & x_i & \geq & 0 & \quad i \in [n] \ . \end{array}$$

a) Which well-known problem from graph-theory is modeled by the LP above, if one adds the integrality (binary) constraints $x_i \in \{0, 1\}$, $i \in [n]$?

b) Develop the dual LP (D_G) of (P_G).

c) Let G be the complete graph, i. e. $E = \{\{i, j\} : i \neq j\}$. Guess optimal solutions for (P_G) and (D_G) (depending on n) and proof their optimality.

d) Find optimal solutions, if in addition to $E = \{\{i, j\} : i \neq j\}$, we demand integrality in P_G for the primal variables and in D_G for the dual variables?

e) Which well known graph-theoretical problem is modeled by (D_G), if one adds integrality constraints for the dual variables?

f) Give a necessary and sufficient condition for G, s. t. the integral versions of (P_G) and (D_G) have the same optimal value?

Solution to problem 1.1

a) Maximum Stable Set ($x_i = 1$ corresponds to adding the node to the stable set, the linear constraints ensure that no two nodes joint by an edge are chosen, the objective is to maximize the number of chosen nodes).

b)

$$\begin{array}{rrcll} \min & \sum_{\{i,j\}\in E} y_{\{i,j\}} & & & \\ & \sum_{i\in[n],\{i,j\}\in E} y_{\{i,j\}} & \geq & 1 & \text{for all } j \in [n] \\ & y & \geq & 0 & \end{array}$$

c) $\frac{n}{2}$. Primal: $x_i = \frac{1}{2}$ for all i is obviously feasible for (P_G); dual: let K a Hamilton circuit and $y_{\{i,j\}} = \frac{1}{2}$ for all $\{i, j\} \in K$. Optimality then follows from duality theory.

d) Primal: 1, since every node is a neighbor of every other. Dual: $\lceil \frac{n}{2} \rceil$ (Choose again a Hamilton circuit K and let $y_{\{i,j\}} = 1$ for every second edge along the circuit. This is an optimal solution, since the optimal fractional value is $\frac{n}{2}$ and therefore the optimal integer solution must have an integer value at least $\lceil n/2 \rceil$.

e) Edge Cover: Find a minimal set of edges $(y_{\{i,j\}=1})$, s. t. all nodes are covered by at least one edge (linear constraints). If there exists an isolated node the problem becomes infeasible (and (P_G) unbounded).

f) Iff G contains no circuits of odd length, which is equivalent to G bipartit (Koenig's Theorem).

Problem 1.2

In the usual sudoku a square of 9×9 fields is given, which is separted in a canonical way into nine 3×3 subsquares. Some of the fields contain prescribed numbers from 1 to 9. The task is to place numbers into the remaining fields, s. t. within each row, each column, and each subsquare all the numbers from 1 to 9 occure exactly once.

Model the sudoku task as

a) a graph coloring problem with precolored nodes

b) an integer linear optimization problem

Solution to problem 1.2

a) Consider a graph $G = (V, E)$ defined as followed: The vertex set is $V = \{v_{11}, \ldots, v_{19}, \ldots, v_{91}, \ldots, v_{99}\}$, where the vertex v_{ij} with $i, j \in [9]$ represents position (i, j) of the sudoku square. The edge set E contains an edge for all pairs of nodes in V, which belong to the same row, columns, or subsquare. Finally, for each presrcibed number within the sudoku square we pre-color the corresponding node accordingly. Now, the task is to find a valid nine-coloring of G.

b) Our model contains $9^3 = 729$ binary variables $\xi_{ijk} \in \{0, 1\},\ i, j, k \in [9]$. Here we want $\xi_{ijk} = 1$, iff the number k should be placed into field (i, j) of the sudoku square and $\xi_{ijk} = 0$ otherwise. Let $S \subset [9]^2$ denote the set of fields (i, j), which are fixed by the entry $k(i, j)$. Moreover, let $Q_l,\ l \in [9]$ denote the l-th subquare. Then the following linear constraints model the sudoku task:

$$\sum_{i=1}^{n} \xi_{ijk} = 1, \quad j, k = 1, \ldots, 9 \tag{1}$$

$$\sum_{j=1}^{n} \xi_{ijk} = 1, \quad i, k = 1, \ldots, 9 \tag{2}$$

$$\sum_{(i,j) \in Q_l} \xi_{ijk} = 1, \quad l = 1, \ldots, 9 \tag{3}$$

$$\sum_{k=1}^{n} \xi_{ijk} = 1, \quad i, j = 1, \ldots, 9 \tag{4}$$

$$\xi_{ij,k(i,j)} = 1, \quad (i, j) \in S \tag{5}$$

Conditions (1),(2),(3) mean, that in each row, column, or subsquare every number may occur exactly once. Condition (4) ensures, that some number is placed in every field, and condition (5) keeps track of the prescribed entries.

Problem 1.3
Show that

a) $\lfloor \frac{\alpha}{\beta} \rfloor \beta \geq \frac{\alpha}{2}$ for $\alpha \geq \beta \geq 1$.

b) the *Euclidean Algorithm*

INPUT:	$\alpha, \beta \in \mathbb{N}$		
OUTPUT:	The greatest common divisor gcd of α and β		
BEGIN	$k \leftarrow 0$, $a_0 \leftarrow \max\{\alpha, \beta\}$, $a_1 \leftarrow \min\{\alpha, \beta\}$		
	REPEAT	$k \leftarrow k+1$, $q_k \leftarrow \lfloor \frac{a_{k-1}}{a_k} \rfloor$, $a_{k+1} \leftarrow a_{k-1} - q_k a_k$	
	UNTIL	$a_{k+1} = 0$	
	gcd $\leftarrow a_k$		
END			

finds a correct solution in at most $\lfloor \log(\alpha \cdot \beta) \rfloor + 1$ iterations.

c) when determining whether an algorithm runs in polynomial time, one may assume for any rational ρ given in the form $\rho = \frac{\alpha}{\beta}$, that α and β are coprime.

d) the Euclidean Algorithm can be extended in such a way that solutions $x_1, x_2 \in \mathbb{Z}$ of the linear Diophantine equation $\alpha \cdot x_1 + \beta \cdot x_2 = \gcd(\alpha, \beta)$ may be computed in polynomial time.

Hint: Construct two suitable progressions of s_k's and t_k's, such that always $\alpha \cdot s_k + \beta \cdot t_k = a_k$. (Why?)

Solution to problem 1.3

a) $2\lfloor \frac{\alpha}{\beta} \rfloor = \lfloor \frac{\alpha}{\beta} \rfloor + \lfloor \frac{\alpha}{\beta} \rfloor \geq (\frac{\alpha}{\beta} - 1) + 1 = \frac{\alpha}{\beta}$, and the desired inequality follows.

b) Correctness: Let $\alpha, \beta \in \mathbb{N}$. If κ is a common divisor, then κ also divides $\alpha - \lfloor \frac{\alpha}{\beta} \rfloor \beta$ (or $\beta \leftarrow \beta - \lfloor \frac{\beta}{\alpha} \rfloor \alpha$). On the other hand: If κ divides β and $\alpha - \lfloor \frac{\alpha}{\beta} \rfloor \beta$, then it also divides α. As a consequence: $\gcd(\alpha, \beta) = \gcd(\beta, \alpha - \lfloor \frac{\alpha}{\beta} \rfloor \beta)$.

The correctness of the algorithm follows iteratively because in the case of the termination one of the numbers is 0 and the other one divides the larger number in the previous step. Therefore it is the gcd.

Duration: From (a) it follows that in every step one of the two numbers is (at least) divided by a factor 2. Hence, $a_k \cdot a_{k+1}$ is (at least) divided by 2 in each step and since this product is always an integer we get $a_k \cdot a_{k+1} = 0$ after at most $\lfloor \log(\alpha \cdot \beta) \rfloor + 1$ steps. However, this means $a_{k+1} = 0$ at that stage.

Altogether, the running time of the algorithm is linear in the number of its iterations and therefore also linear in terms of the encoding-lengths.

c) Because of the polynomiality of the Euclidean Algorithm, non-coprime fractions can be converted efficiently into coprime ones.

d) *Extended Euclidean Algorithm*

INPUT:	$\alpha, \beta \in \mathbb{N}$
OUTPUT:	The greatest common divisor gcd of α and β
BEGIN	$k \leftarrow 0, \quad a_0 \leftarrow \max\{\alpha, \beta\}, \quad a_1 \leftarrow \min\{\alpha, \beta\},$
	$s_0 \leftarrow 1, \quad s_1 \leftarrow 0, \quad t_0 \leftarrow 0, \quad t_1 \leftarrow 1$
	REPEAT $\quad k \leftarrow k+1, \quad q_k \leftarrow \lfloor \frac{a_{k-1}}{a_k} \rfloor, \quad a_{k+1} \leftarrow a_{k-1} - q_k a_k$
	$\qquad s_{k+1} \leftarrow s_{k-1} - q_k s_k, \quad t_{k+1} \leftarrow t_{k-1} - q_k t_k$
	UNTIL $\quad a_{k+1} = 0$
	$\text{gcd} \leftarrow a_k$
END	

With the chosen initialization it holds that

$$a_0 = \alpha = 1 \cdot \alpha + 0 \cdot \beta = s_0 \cdot \alpha + t_0 \cdot \beta$$

and

$$a_1 = \beta = 0 \cdot \alpha + 1 \cdot \beta = s_1 \cdot \alpha + t_1 \cdot \beta$$

The configuration of the s_k and t_k is analogous to the configuration of the a_k. This means that $a_k = s_k \cdot \alpha + t_k \cdot \beta$ always applies. At the end of the algorithm we have $\gcd(\alpha, \beta) = a_k$, which shows that the algorithm is correct.

Problem 1.4

a) Suppose $P \subset \mathbb{R}^2$ is given by the two inequalities $-\sqrt{2}x_1 + x_2 \leq 0$ und $x_1 - \sqrt{2}x_2 \leq 0$ ($\mathcal{H}$-representation).

(i) Show $P = \text{pos}\{(1, \sqrt{2})^T, (\sqrt{2}, 1)^T\}$ ($\mathcal{V}$-representation).

(ii) The integer hull of P is defined as $P_I := \text{conv}(P \cap \mathbb{Z}^2)$. Show $P_I = \{0\} \cup \text{int}(P)$. (Is P_I a polyhedron?)

b) Suppose $Q = \{(-\frac{1}{2}, -\frac{1}{2})^T\} + \text{pos}\{(1, \sqrt{2})^T, (\sqrt{2}, 1)^T\}$. Show: The integer hull $Q_I = \text{conv}(P \cap \mathbb{Z}^2)$ of Q has an infinite number of edges and vertices.

Use without proof: For every line in $\mathbb{R}^2$ with an irrational slope there exist integer points arbitrarily close to it.

Solution to problem 1.4

a) (i) elementary, by calculation

(ii) Obvious: $0 \in P_I$ and no other point of the boundary of P is in P_I because this could only be if another integer point lays on the boundary, which is not the case because of the irrational slope of the tracing lines. Assume $x \in \text{int}\, P \setminus P_I$, w. l. o. g. x rational (because $\mathbb{Q}^2$ is dense within $\mathbb{R}^2$), with the representation $x = (\frac{a}{b}, \frac{c}{d})$ with $a, b, c, d \in \mathbb{Z}$ follows $b \cdot d \cdot x \in P \cap \mathbb{Z}^2$ and so $x \in \text{conv}\{0, b \cdot d \cdot x\} \subset P_I$, in contradiction to the assumption.

b) Obviously 0 is a vertex of Q_I. Assume Q_I has only finitely many edges (and vertices). Let $p_0 = 0$. Now, the vertex next to p_0 in $\mathbb{Z}^2_+$ which is closer to $l = \{(-\frac{1}{2}, -\frac{1}{2})^T + \lambda(1, \sqrt{2})^T : \lambda \geq 0\}$ is denoted as p_1 and iteratively the vertex different from p_{i-1} on an edge containing p_i as p_{i+1}. Because of the finite number of vertices (and Q_I is unbounded) there must be $n \in \mathbb{N}$ such that p_n has only one neighbor (p_{n-1}). Consider the edges k parallel to l through p_n (notice that p_n is not on l because there is at most one rational point on l which is $\left(-\frac{1}{2}, -\frac{1}{2}\right)^T$). With the given hint one sees that there exists an integer point between k and l. So k can not be an edge of P_I in contradiction to the choice of p_n.

Concerning the hint: First we show that for every line which contains an integer point (w. l. o. g. $(0,0)^T$) there are more integer points arbitrary close to it. We can represent the line (w. l. o. g.) as $x_2 = rx_1$, with r irrational. From number theory it is known: for all $r \in \mathbb{R}$ exists $a, b \in \mathbb{Z}$ such that $|r - \frac{b}{a}| < \frac{1}{a^2}$. Consider $(a,b)^T \in \mathbb{Z}^2$ with $|r - \frac{b}{a}| < \frac{1}{a^2}$ and $0 < \frac{1}{a} < \varepsilon$ than it applies for (a, x_2) on the line $|x_2 - b| = |ra - b| < \frac{1}{a} < \varepsilon$. So the distance of $(a,b)^T$ to the line is less than ε.

Now, w. l. o. g. we may assume that the line is represented as $x_2 = rx_1 + t$ with $t \in (0,1)$. Choose $c, d \in \mathbb{Z}$ such that $|\frac{c}{d} - t| < \frac{\varepsilon}{2}$ and $(a,b)^T \in \mathbb{Z}^2$ such that $|ra - b| < \frac{\varepsilon}{2}$. It follows $|(ra + t) - (b + \frac{c}{d})| = \delta < \varepsilon$, i.e. the rational point $(a, b + \frac{c}{d})$ is sufficiently close to (wlog underneath) the line. Let $n \in \mathbb{N}$ such that $(n-1)\delta < 1 - \frac{c}{d} < n\delta$ so $1 - \frac{c}{d} < n\delta < 1 - \frac{c}{d} + \delta$ it follows $(rna + t) - (nb + 1) = n\delta + \frac{c}{d} - 1 \in (0, \delta)$, i.e. $(na, nb + 1) \in \mathbb{Z}^2$ is maximal ε away from the line.