Discrete Optimization (MA 3502), WiSe 2012/13

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Supplementary Problem Sheet 1

Supplementary Problem 1.1
For $k \in \mathbb{N}_0 \cup \{-1\}$ let ϕ_k denote the k-th Fibonacci-number (i.e. $\phi_{-1} = 1, \phi_0 = 0$, and $\phi_k = \phi_{k-1} + \phi_{k-2}$, $k \geq 1$) and

$$T^k := \{x \in \mathbb{R}^2 : \phi_{2k}x_1 + \phi_{2k+1}x_2 \leq \phi_{2k+1}^2 - 1 \ , \ x_1, x_2 \geq 0\}$$

a series of triangles built from that numbers. show: The integer hull T_I^k of T^k has vertices

$$V_j = (\phi_{2j}, \phi_{2k+1} - \phi_{2j-1})^T, \ j = 0, \ldots, k+1, \text{ and } V_{k+2} = (0,0)^T,$$

i.e. T_I^k has $k+3$ vertices and as many edges (facets in dimension 2).

Use without proof:

$$\phi_{2k+1}^2 = \phi_{2k}\phi_{2k+2} + 1 \text{ and } \phi_{2k}^2 = \phi_{2k-1}\phi_{2k+1} - 1 \text{ for all } k \geq 0, \quad (1)$$
$$\phi_{2k}\phi_{2j+1} - \phi_{2j}\phi_{2k+1} \geq 0 \text{ for all } k \geq 0 \text{ and } 0 \leq j \leq k, \quad (2)$$
$$\phi_{2k+1}\phi_{2j-1} - \phi_{2k}\phi_{2j} \geq 1 \text{ for all } k \geq 0 \text{ and } 0 \leq j \leq k+1. \quad (3)$$

Remark: Use the following puzzle-pieces in your proof:

(i) The edges $E_j = \text{conv}\{V_{j-1}, V_j\}$, $j \in [k+2]$ and $E_{k+3} = \text{conv}\{V_{k+2}, V_0\}$ are located on the lines

$$\phi_{2j-2}x_1 + \phi_{2j-1}x_2 = \phi_{2k+1}\phi_{2j-1} - 1, \text{ for } j \in [k+1],$$

$y = 0$ for $j = k+2$, and $x = 0$ for $j = k+3$, respectively.

(ii) The edges E_j, $j \in [k+1]$ have non-positive slopes, which strictly decrease with raising j.

(iii) $V_j \in T^k$ for $0 \leq j \leq k+2$.

(iv) Any point $x \in \mathbb{Z}_+^2$, not conatianed in the convex hull of the V_j's is also not in T^k. Here, consider $j \in [k+1]$ such that $\phi_{2j-2} \leq x_1 \leq \phi_{2j}$ and show that

$$\phi_{2j-2}x_1 + \phi_{2j-1}x_2 \geq \phi_{2k+1}\phi_{2j-1}.$$

Solving this inequality for x_2 than maybe used to show $x \notin T^k$.

Submission: by Monday, 14:00 in the week following the exercise class, in the metal cupboard at the beginning of the M9-finger or in my office.

Please submit your work in groups.