Technische Universität München
Zentrum Mathematik

Discrete Optimization (MA 3502), WiSe 2012/13

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Problem Sheet 1

Problem 1.1
Let $G = (V, E)$ a graph with node set $V = [n]$ and $n \geq 2$. Consider the following LP:

$$\begin{array}{llll} \max & \sum_{i\in[n]} x_i & & \\ & x_i + x_j \leq 1 & \text{for all } \{i,j\} \in E\ , & (P_G) \\ & x_i \geq 0 & i \in [n]\ . & \end{array}$$

a) Which well-known problem from graph-theory is modeled by the LP above, if one adds the integrality (binary) constraints $x_i \in \{0, 1\}$, $i \in [n]$?

b) Develop the dual LP (D_G) of (P_G).

c) Let G be the complete graph, i. e. $E = \{\{i, j\} : i \neq j\}$. Guess optimal solutions for (P_G) and (D_G) (depending on n) and proof their optimality.

d) Find optimal solutions, if in addition to $E = \{\{i, j\} : i \neq j\}$, we demand integrality in P_G for the primal variables and in D_G for the dual variables?

e) Which well known graph-theoretical problem is modeled by (D_G), if one adds integrality constraints for the dual variables?

f) Give a necessary and sufficient condition for G, s. t. the integral versions of (P_G) and (D_G) have the same optimal value?

Problem 1.2
In the usual sudoku a square of 9×9 fields is given, which is separted in a canonical way into nine 3×3 subsquares. Some of the fields contain prescribed numbers from 1 to 9. The task is to place numbers into the remaining fields, s. t. within each row, each column, and each subsquare all the numbers from 1 to 9 occure exactly once.

Model the sudoku task as

a) a graph coloring problem with precolored nodes

b) an integer linear optimization problem

Please turn over!

Problem 1.3
Show that

a) $\lfloor \frac{\alpha}{\beta} \rfloor \beta \geq \frac{\alpha}{2}$ for $\alpha \geq \beta \geq 1$.

b) the *Euclidean Algorithm*

INPUT:	$\alpha, \beta \in \mathbb{N}$	
OUTPUT:	The greatest common divisor gcd of α and β	
BEGIN	$k \leftarrow 0$, $a_0 \leftarrow \max\{\alpha, \beta\}$, $a_1 \leftarrow \min\{\alpha, \beta\}$	
	REPEAT	$k \leftarrow k+1$, $q_k \leftarrow \lfloor \frac{a_{k-1}}{a_k} \rfloor$, $a_{k+1} \leftarrow a_{k-1} - q_k a_k$
	UNTIL	$a_{k+1} = 0$
	gcd $\leftarrow a_k$	
END		

finds a correct solution in at most $\lfloor \log(\alpha \cdot \beta) \rfloor + 1$ iterations.

c) when determining whether an algorithm runs in polynomial time, one may assume for any rational ρ given in the form $\rho = \frac{\alpha}{\beta}$, that α and β are coprime.

d) the Euclidean Algorithm can be extended in such a way that solutions $x_1, x_2 \in \mathbb{Z}$ of the linear Diophantine equation $\alpha \cdot x_1 + \beta \cdot x_2 = \gcd(\alpha, \beta)$ may be computed in polynomial time.

Hint: Construct two suitable progressions of s_k's and t_k's, such that always $\alpha \cdot s_k + \beta \cdot t_k = a_k$. (Why?)

Problem 1.4
a) Suppose $P \subset \mathbb{R}^2$ is given by the two inequalities $-\sqrt{2}x_1 + x_2 \leq 0$ und $x_1 - \sqrt{2}x_2 \leq 0$ ($\mathcal{H}$-representation).

(i) Show $P = \text{pos}\{(1, \sqrt{2})^T, (\sqrt{2}, 1)^T\}$ ($\mathcal{V}$-representation).

(ii) The integer hull of P is defined as $P_I := \text{conv}(P \cap \mathbb{Z}^2)$. Show $P_I = \{0\} \cup \text{int}(P)$. (Is P_I a polyhedron?)

b) Suppose $Q = \{(-\frac{1}{2}, -\frac{1}{2})^T\} + \text{pos}\{(1, \sqrt{2})^T, (\sqrt{2}, 1)^T\}$. Show: The integer hull $Q_I = \text{conv}(P \cap \mathbb{Z}^2)$ of Q has an infinite number of edges and vertices.

Use without proof: For every line in $\mathbb{R}^2$ with an irrational slope there exist integer points arbitrarily close to it.

Submission: by Monday, 14:00 in the week following the exercise class, in the metal cupboard at the beginning of the M9-finger or in my office.

Please submit your work in groups.