**Technische Universität München**
Zentrum Mathematik

# Discrete Optimization (MA 3502), WiSe 2012/13

Prof. Dr. P. Gritzmann, Dr. René Brandenberg

## Problem Sheet 2

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### Problem 2.1

Consider the following sets of vectors in $\mathbb{R}^2$:

$$\text{(i)}\ \left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\} \qquad \text{(ii)}\ \left\{\begin{pmatrix}2\\4\end{pmatrix},\begin{pmatrix}3\\6\end{pmatrix}\right\} \qquad \text{(iii)}\ \left\{\begin{pmatrix}5\\1\end{pmatrix}\right\}$$

$$\text{(iv)}\ \left\{\begin{pmatrix}3\\2\end{pmatrix},\begin{pmatrix}4\\3\end{pmatrix}\right\} \qquad \text{(v)}\ \left\{\begin{pmatrix}2\\1/2\end{pmatrix},\begin{pmatrix}1/2\\2\end{pmatrix}\right\}$$

a) Which sets generate lattices in $\mathbb{R}^2$?

b) Which sets generate sublattices of $\mathbb{Z}^2$?

Let $L_1, \ldots, L_5$ be the lattices generated by $(i), \ldots, (v)$ respectively. Consider the partially ordered set (poset) consisting of $L_1, \ldots, L_5$ ordered by inclusion. (See „Einführung in die Diskrete Mathematik“ for details on posets)

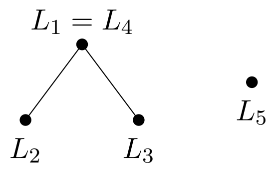
c) Draw the Hasse Diagram of this partially ordered set.

*Solution to problem 2.1*

a) Sets (i), (iv) and (iv) generate lattices in $\mathbb{R}^2$. Sets (ii) and (iii) do not, since the corresponding sets $L_1$ and $L_2$ each lie in a 1-dimensional subspace, from which we cannot choose a set of vectors which form a basis of $\mathbb{R}^2$.

b) Sets (i), (ii), (iii) and (iv) generate sublattices of $\mathbb{Z}^2$ ($L_2$ is also generated by the linearly independent set $\{\binom{1}{2}\}$). Set (v) does not, since $L_5$ is not a subset of $\mathbb{Z}^2$.

c)




### Problem 2.2

Give an example of a matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{Z}^{2\times 2}$ and a vector $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{Z}^2$, s. t. the equations $a_{11}x_1 + a_{12}x_2 = b_1$ and $a_{21}x_1 + a_{22}x_2 = b_2$ both have a solution $x \in \mathbb{Z}^2$, but the system of equations $Ax = b$ does not.

*Solution to problem 2.2*

One example would be $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $b = \binom{1}{0}$

**Problem 2.3**

Let $G$ be a graph with $n$ vertices and $m$ edges. The *incidence matrix* $S_G \in \{0,1\}^{n\times m}$ has rows corresponding to vertices of $G$, columns corresponding to edges, and an entry 1 whenever the corresponding vertex and edge are incident (0 otherwise) – compare Problem 1.1.

a) Suppose that $G$ is connected. Show that $\operatorname{rank}(S_G) = \begin{cases} n-1 & , G \text{ bipartite} \\ n & , \text{otherwise} \end{cases}$

b) Consider a subgraph $K$ of $G$ with $|V(K)| = n$, $|E(K)| = n-1$ and $S_K$ the incidence matrix of $K$. We denote by $S_{K,i}$ the matrix $S_K$ where the $i$-th row has been deleted. Show that:

$$K \text{ is a tree } \Leftrightarrow \forall i \in [n] : \ |\det(S_{K,i})| = 1 \ \Leftrightarrow \ \exists i \in [n] : \ |\det(S_{K,i})| = 1$$

c) Consider a connected subgraph $K$ of $G$ with $|V(K)| = n$, $|E(K)| = n$ and $S$ the incidence matrix of $K$. Show that:

$$K \text{ contains an odd cycle} \Leftrightarrow |\det(S)| = 2$$

*Solution to problem 2.3*

a) Consider the columns of $S_G$ corresponding to the edges of a spanning tree $T$ of $G$. These form an $n \times (n-1)$ submatrix of $S_G$. By taking a leaf as the first vertex $v_1$, and its incident edge as the first vertex $e_1$, we can ensure that the matrix has a 1 in the top left corner, and that the rest of the first row consists of 0s. Thus the first column will be linearly independent of the other $n-2$ columns. Deleting this column, the first row has only 0 entries, so we also delete this row. This corresponds to deleting $v_1$ and $e_1$ from $T$. Since this leaves us with a smaller tree, we can continue in the same way. We thus show inductively that all the columns of the submatrix are linearly independent, and so $S_G$ has rank at least $n-1$.

   $T$ induces a unique bipartition of the vertices of $G$. Suppose $G$ respects this bipartition. Then we can partition the rows of $G$ into index sets $I_1, I_2$ such that each column has a 1 in a row from $I_1$ and a 1 in a row from $I_2$. But then in any linear combination of column vectors, the sum over rows from $I_1$ and rows from $I_2$ will be identical. Hence, for example, the standard basis vector $u^1$ cannot be expressed as a linear combination of columns in $S_G$. Thus the columns do not span $\mathbb{R}^n$, and so the matrix does not have rank $n$. Together with the proof above, this shows that if $G$ is bipartite, then $S_G$ has rank $n-1$.

   On the other hand, if $G$ is not bipartite, then there exists an edge which does not respect the bipartition induced by $T$. The corresponding column then has two 1s in rows from the same partition set $I_j$. Since the sum over the partition sets are then not equal (2 and 0 in some order), this column vector cannot be expressed as a linear combination of the column vectors corresponding to the edges of $T$. Thus these $n$ columns together form a linearly independent set, and $S_G$ has rank $n$.

b) We prove „$Katree \Rightarrow \forall i : |\det(S_{K,i})| = 1$“ by induction. The base case $n = 2$ is easy.

For the inductive step, let $v$ be the vertex corresponding to the row $i$ to be deleted. We replace $v$ by $c$ distinct vertices, where $c$ is the number of components of $K - v$, each one joined to a distinct neighbour of $v$. The incidence matrix of the new graph has $n - 1$ columns and $n + c - 1$ rows, and $S_{K,i}$ is identical to this matrix with the $c$ new rows deleted. Moreover, the rows and columns of the new matrix can be re-ordered so that it is in diagonal block form. Then, once we have deleted the $c$ rows, the determinant is the product of the determinants of each block, which by the inductive hypothesis are all $\pm 1$.

If $c = 1$, we cannot apply the inductive hypothesis, as we have just one block which is exactly $S_{K,i}$. In this case, $v$ is a leaf, so let $j$ be the row corresponding to its neighbour. We note that the column corresponding to the edge between these two vertices has just one non-zero entry in $S_{K,i}$, which lies in column $j$. Thus $|\det(S_{K,i})| = |\det((S_{K,i})_j| = 1$, where the last equality follows by the inductive hypothesis.

If $K$ is not a tree, then it contains a cycle. If this cycle is even, then the corresponding columns are linearly dependent (even after deleting any row), and thus the matrix $S_{K,i}$ is singular, so $\det(S_{K,i}) = 0$. If the cycle is odd, and the row deleted corresponds to a vertex in the cycle, then there is a component of the graph which is a tree, with $k$ vertices, say. Then the corresponding $k$ rows of the matrix $S_{K,i}$ have non-zero entries in only $k - 1$ columns, and thus form a linearly dependent set. Thus $S_{K,i}$ is singular, so $\det(S_{K,i}) = 0$.

If the cycle is odd, and the row deleted doesn't correspond to a vertex in the cycle, then the submatrix $S'$ corresponding to the cycle has determinant 2 (as will be proved in c). By arranging the matrix in upper-triangular block form, where one block is $S'$, then $\det(S')$ is a factor of $\det(S_{K,i})$, and since $S_{K,i}$ is integral, all other factors are certainly integral, and in particular $\det(S_{K,i})$ is even.

Thus in all cases $\det(S_{K,i}) \neq \{-1, 1\}$.

Alternatively, one may choose node $i$ as a root of the tree and sort the other nodes and edges with help of a BFS/DFS-algorithm. It is quite easy to see, that after permuting the rows and columns of $S_{K,i}$ accordingly, we obtain an upper triangular matrix with only 1's along the diagonal.

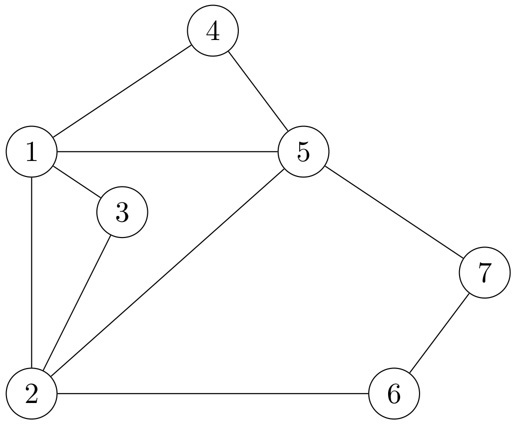
c) If $K$ is a cycle on $k$ vertices, then we can rearrange the incidence matrix to have 1s on the main diagonal, 1s on the subdiagonal, a 1 in the top right corner and 0s elsewhere. It is easy to check that this matrix has determinant 2.

Now if $K$ is not a cycle, then it contains a leaf. Deleting this leaf and the incident edge does not change the determinant (up to sign) of the incidence matrix. We can therefore delete leaves until the remaining graph is exactly an odd cycle, whose incidence matrix has determinant $\pm 2$. Since deleting leaves does not change the absolute value of the determinant, the determinant of the original incidence matrix is also $\pm 2$.

**Problem 2.4**
Let $K_n = ([n], \binom{V}{2})$ the complete graph on $n$ vertices and $c : E \to \mathbb{N}$ a edge-weight function. We say „$c$ fulfills the triangle inequality“ if $c(\{i,j\}) + c(\{j,k\}) \geq c(\{i,k\})$ for all $i, j, k \in [n]$.

a) Show: Every graph, having a minimal degree $\delta(G) \geq 2$ includes a simple cycle (a closed path).

b) A trail in a graph is a set of consecutive edges. A walk in a graph is a trail without edge repetitions. (Hence a walk is a path, if there are no node repetitions either). An eulerian cycle (or eulerian tour) is a cycle (a closed walk) using each *edge* of the graph exactly once.
Find an eulerian cycle in the following graph




c) Show: Any connected graph $G = (V, E)$, s.t. $\deg(v) \in 2\mathbb{N}$ for all $v \in V$ contains an eulerian cycle. Can you give an (efficient) algorithm for the construction of such a cycle in such graphs? What happens if $G$ may be a multi-graph?

d) Suppose $T = ([n], E)$ is a minimal spanning tree of $G$ with respect to $c$. Show: $c(T)$ is a lower bound for the length of any TSP tour in $G$.

e) Suppose $C = (e_1, e_2, \ldots, e_m)$ with $e_1, \ldots, e_m \in \binom{V}{2}$ is a cycle containing all nodes of $V$ of length $c(C) = \sum_{i\in[m]} c(e_i)$. Show: If $c$ fulfills the triangle inequality, then there exists a Traveling Salesman tour (a simple cycle containing all vertices of the graph) in $K_n$, of length at most $c(C)$. How can we construct the TSP tour from the eulerian tour?

f) Use the above to give an algorithm, which constructs a TSP tour of length at most two times the length of a minimal TSP tour. (Such an algorithm is called a *factor 2-approximation*).

*Solution to problem 2.4*

a) Let $G = (V, E)$ and $P$ a longest (in the number of edges contained) (open) path in $G$. Let $w$ the last node of the path. Since $\deg(w) \geq 2$ there exists an edge $e = \{w, w'\} \in E$ not belonging to the path. Assuming $w'$ would not belong to $P$ would lead to the contradiction that $P$ could be extended by $e$ to a longer path. Thus $w'$ must belong to $P$, which means that the part of $P$ connecting $w$ and $w'$ is closed to a simple cycle using $e$.

b) Since $\deg(v) \in 2\mathbb{N}$ for all $v \in V$ means $\deg(v) \geq 2N$ for all $v \in V$, we know from (a) that $G$ contains a simple cycle. Removing all edges belonging to the cycle from $G$ and afterwards all nodes having degree 0, the reduced graph still posseses the property $\deg(v) \in 2\mathbb{N}$ for all its nodes. Hence $G$ may successively be decomposited in edge-disjoint simple cycles. Since $G$ connected, we may join those cycles to one eulerian tour. If $G$ may be a multi-graph or not has no influence on this solution.

c) If we remove an arbitrary edge from a TSP tour, it gets a Hamiltonian path and therefore a spanning tree of $G$. Hence the length of a minimal spanning tree $c(T)$ is a lower bound for the length of the TSP tour.

d) Let $(v_1, \dots, v_n)$ denote the nodes of $K_n$ enumerated along their first visit by $C$ (starting at an arbitrary node $v_1$). Then $(v_1, \dots, v_n)$ and the direct edges $\{v_i, v_{i+1}\}$ and $\{v_1, v_n\}$ describe a TSP tour. Since the length of each of the edges $\{v_i, v_{i+1}\}$ is (because of the triangle inequality) at most the length of the path from $v_i$ to $v_{i+1}$ along the eulerian tour, the constructed TSP tour has length at most $c(C)$.

e) Starting at any node $v_1 \in V$ we obtain an order $(v_1, \dots, v_n)$ of first appearences of all nodes along the cycle $C$. By multiple application of the triangle inequality, the Hamiltonian cycle along the direct edges $\{v_i, v_{i+1}\}$ and $\{v_n, v_1\}$ is not longer than $c(C)$.

f) If we compute a minimal spanning tree $T$ and double all its edges, we get a multi graph $2T$ fulfilling $\deg(v)$ even for all its nodes. Because of (c), there exists an eulerian cycle through $2T$ and because of (d,e) we may short cut it to a TSP tour $H$ of length $c(T) \leq c(H*) \leq c(H) \leq 2c(T)$, where $H^*$ denotes the minimal TSP tour.