Technische Universität München
Zentrum Mathematik

Discrete Optimization (MA 3502), WiSe 2012/13

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Supplementary Problem Sheet 2

Problem 2.1

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ a pointed polyhedron, where $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$. Moreover, may α be the maximum of all absolut values of the subdeterminantes of the extended coefficiant matrix (A, b). Show:

a) If $V \subset P$ is the vertex set of P, then $V \subset \alpha[-1, 1]^n$.

b) The set of unbounded directions $Z \subset \mathbb{Q}^n$ in the $\mathcal{V}$-representation of $P = \mathrm{conv}(V) + \mathrm{pos}(Z)$ may be chosen such that $Z \subset [-1, 1]^n$ and $\alpha Z \in \mathbb{Z}^n$, and therefore also such that $Z \subset \alpha[-1, 1]^n \cap \mathbb{Z}^n$.

c) There exists $W \subset \alpha(n + 1)[-1, 1]^n \cap \mathbb{Z}^n$, such that $P_I = \mathrm{conv}(W) + \mathrm{pos}(Z)$ (compare to Lemma 1.3.2).

   (Why is it possible to conclude that $P \cap \mathbb{Z}^n \neq \emptyset \Leftrightarrow P \cap \mathbb{Z}^n \cap \alpha(n + 1)[-1, 1]^n \neq \emptyset$ and why could that be important?)

d) From Hadamard's-Inequality (see below) it follows $\alpha \leq n^{\frac{n}{2}} \cdot M^n$, where M is the maximum of all absolut values within (A, b), i.e.

$$M := \max\{\max\{|a_{ij}| : i \in [m], j \in [n]\}, \max\{|\beta_i| : i \in [m]\}\}.$$

e) The upper bound in (d) is attained by the Hadamard-Matrices (see below) H_n, i.e. $|\det(H_n)| = n^{\frac{n}{2}}$.

f) Hadamard-matrices exist in infinitly many dimensions.

   Hint: To show this, first give a Hadamard matrix of order 2 and then design a rule to construct a Hadamard matrix of order $2n$ from one of order n.

Hadamard's-Inequality: If $C \in \mathbb{R}^{n\times n}$ with column vectors $c_1, \ldots, c_n$, then $|\det(C)| \leq \prod_{i=1}^n \|c_i\|_2$.

Hadamard-matrix: A Hadamard-matrix of order n is an $(n \times n)$-matrix $H_n \in \{-1, +1\}^{n\times n}$, such that its column (and row) vectors are pairwise orthogonal.

Submission: by Monday, 14:00 in the week following the exercise class, in the metal cupboard at the beginning of the M9-finger or in my office.

Please submit your work in groups.