Technische Universität München
Zentrum Mathematik

Discrete Optimization (MA 3502), WiSe 2012/13

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Problem Sheet 2

Problem 2.1
Consider the following sets of vectors in $\mathbb{R}^2$:

$$\text{(i)}\ \left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\} \qquad \text{(ii)}\ \left\{\begin{pmatrix}2\\4\end{pmatrix},\begin{pmatrix}3\\6\end{pmatrix}\right\} \qquad \text{(iii)}\ \left\{\begin{pmatrix}5\\1\end{pmatrix}\right\}$$

$$\text{(iv)}\ \left\{\begin{pmatrix}3\\2\end{pmatrix},\begin{pmatrix}4\\3\end{pmatrix}\right\} \qquad \text{(v)}\ \left\{\begin{pmatrix}2\\1/2\end{pmatrix},\begin{pmatrix}1/2\\2\end{pmatrix}\right\}$$

a) Which sets generate lattices in $\mathbb{R}^2$?

b) Which sets generate sublattices of $\mathbb{Z}^2$?

Let $L_1, \dots, L_5$ be the lattices generated by $(i), \dots, (v)$ respectively. Consider the partially ordered set (poset) consisting of $L_1, \dots, L_5$ ordered by inclusion. (See „Einführung in die Diskrete Mathematik“ for details on posets)

c) Draw the Hasse Diagram of this partially ordered set.

Problem 2.2
Give an example of a matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{Z}^{2\times 2}$ and a vector $b = \binom{b_1}{b_2} \in \mathbb{Z}^2$, s. t. the equations $a_{11}x_1 + a_{12}x_2 = b_1$ and $a_{21}x_1 + a_{22}x_2 = b_2$ both have a solution $x \in \mathbb{Z}^2$, but the system of equations $Ax = b$ does not.

Problem 2.3
Let G be a graph with n vertices and m edges. The *incidence matrix* $S_G \in \{0,1\}^{n\times m}$ has rows corresponding to vertices of G, columns corresponding to edges, and an entry 1 whenever the corresponding vertex and edge are incident (0 otherwise) – compare Problem 1.1.

a) Suppose that G is connected. Show that $\operatorname{rank}(S_G) = \begin{cases} n-1 & , G \text{ bipartite} \\ n & , \text{otherwise} \end{cases}$

b) Consider a subgraph K of G with $|V(K)| = n$, $|E(K)| = n-1$ and S_K the incidence matrix of K. We denote by $S_{K,i}$ the matrix S_K where the i-th row has been deleted. Show that:

$$K \text{ is a tree } \Leftrightarrow \forall i \in [n]: \ |\det(S_{K,i})| = 1 \ \Leftrightarrow \ \exists i \in [n]: \ |\det(S_{K,i})| = 1$$

c) Consider a connected subgraph K of G with $|V(K)| = n$, $|E(K)| = n$ and S the incidence matrix of K. Show that:

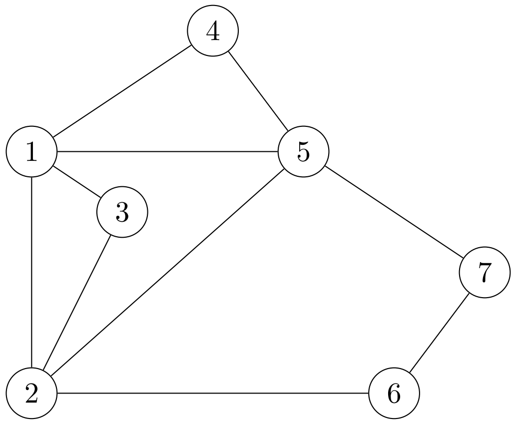
$$K \text{ contains an odd cycle} \Leftrightarrow |\det(S)| = 2$$

Please turn over!

Problem 2.4
Let $K_n = ([n], \binom{V}{2})$ the complete graph on n vertices and $c : E \to \mathbb{N}$ a edge-weight function. We say „c fulfills the triangle inequality" if $c(\{i,j\}) + c(\{j,k\}) \geq c(\{i,k\})$ for all $i, j, k \in [n]$.

a) Show: Every graph, having a minimal degree $\delta(G) \geq 2$ includes a simple cycle (a closed path).

b) A trail in a graph is a set of consecutive edges. A walk in a graph is a trail without edge repetitions. (Hence a walk is a path, if there are no node repetitions either). An eulerian cycle (or eulerian tour) is a cycle (a closed walk) using each *edge* of the graph exactly once.
Find an eulerian cycle in the following graph



c) Show: Any connected graph $G = (V, E)$, s. t. $\deg(v) \in 2\mathbb{N}$ for all $v \in V$ contains an eulerian cycle. Can you give an (efficient) algorithm for the construction of such a cycle in such graphs? What happens if G may be a multi-graph?

d) Suppose $T = ([n], E)$ is a minimal spanning tree of G with respect to c. Show: $c(T)$ is a lower bound for the length of any TSP tour in G.

e) Suppose $C = (e_1, e_2, \ldots, e_m)$ with $e_1, \ldots, e_m \in \binom{V}{2}$ is a cycle containing all nodes of V of length $c(C) = \sum_{i \in [m]} c(e_i)$. Show: If c fulfills the triangle inequality, then there exists a Traveling Salesman tour (a simple cycle containing all vertices of the graph) in K_n, of length at most $c(C)$. How can we construct the TSP tour from the eulerian tour?

f) Use the above to give an algorithm, which constructs a TSP tour of length at most two times the length of a minimal TSP tour. (Such an algorithm is called a *factor 2-approximation*).

Submission: by Monday, 14:00 in the week following the exercise class, in the metal cupboard at the beginning of the M9-finger or in my office.

Please submit your work in groups.