Discrete Optimization (MA 3502), WiSe 2012/13

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Problem Sheet 3

Problem 3.1
Determine

a) all integer solutions of the equation $30 \cdot x_1 + 42 \cdot x_2 + 105 \cdot x_3 = 48$.

b) all $\alpha \in \mathbb{Z}$ such that the Linear Diophantine Equation $30 \cdot x_1 + 42 \cdot x_2 + 105 \cdot x_3 = \alpha$ is solvable, but for all possible solutions $x_i \neq 1$, $i \in \{1, 2, 3\}$, holds.

Solution to problem 3.1

a) If you do not find a solution of the linear Diophantine equation on closer inspection you can still avoid the sophisticated procedure to solve simultaneous equations:

Because of $\gcd(30, 42) = 6$ we could change $30 \cdot x_1 + 42 \cdot x_2$ to $6\hat{x}$ and solve the LDE $6\hat{x} + 105x_3 = 48$ by closer inspection, e.g. $\hat{x} = 8, x_3 = 0$. This means $6\hat{x} = 48$ and in the second step we solve the LDE $30 \cdot x_1 + 42 \cdot x_2 = 48$ perhaps with $x_1 = -4, x_2 = 4$.
Now, for finding all solutions it suffices to find to l. i. solutions of the homogeneous system $30 \cdot x_1 + 42 \cdot x_2 + 105 \cdot x_3 = 0$, which is easy as we can just set one entry 0 and the others to the factor mssing in the collection of prime factors (with alternating signs). For example, let $x_1 = 0$, the we choose $x_2 = 5$ and $x_3 = -2$ as the collection of prime factors is $\{2, 3, 5, 7\}$. Another l. i. solution is $x_1 = 7$ and $x_3 = -2$. Altogeher: $x = (-4, 4, 0)^T + \lambda_1(0, 5, -2)^T + \lambda_2(7, 0, -2)^T$.

b) At first we notice that $30 \cdot x_1 + 42 \cdot x_2 + 105 \cdot x_3 = \alpha$ is solvable, iff $\gcd(30, 42, 105)|\alpha$. So it has to be $\alpha \in 3\mathbb{Z}$ or in other terms $\alpha \equiv 0 \mod 3$.
A solution for the LDE with $x_1 = 1$ exists, iff there is a solution for $42 \cdot x_2 + 105 \cdot x_3 = \alpha - 30$ and this solution exists, iff $\gcd(42, 105) = 21|(\alpha - 30)$. This yields to $\alpha \not\equiv 2 \mod 7$. For $x_2 = 1$ and $x_3 = 1$ we yield $\alpha \not\equiv 2 \mod 5$ and $\alpha \not\equiv 1 \mod 2$.
In a slightly closer form: Forall $\alpha \in 6\mathbb{Z}$ with $\alpha \not\equiv 2 \mod 5$ and $\alpha \not\equiv 2 \mod 7$ (and just for these) exists a solution for the LDE $30 \cdot x_1 + 42 \cdot x_2 + 105 \cdot x_3 = \alpha$, with none of the possible solutions has coordinate $x_i = 1$, $i \in \{1, 2, 3\}$.

Problem 3.2
Consider the linear system of equations

$$Ax = b, \quad A = \begin{pmatrix} 4 & 2 & 2 & 2 \\ 1 & 3 & 4 & 5 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{Z}^2.$$

Determine all vectors b, such that there exists an integer solution x of the system and for each such b the set $\{x \in \mathbb{Z}^4 : Ax = b\}$.

Solution to problem 3.2

a) To determine the possible $b \in \mathbb{Z}^2$, so that the system is solvable, we first compute the Hermite normal form HNF(A) of A and the associated unimodular transformationmatrix U. After that we always label the actual j-th column with (j).

Row 1: As Operations we use (in this order) $(1) := (1)-2(2)$, $(3) := (3)-1(2)$, $(4) := (4)-1(2)$ and the swap of column (1) and (2). All in all we get the unimodular matrix

$$U_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -2 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$A' = AU_1 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 3 & -5 & 1 & 2 \end{pmatrix}.$$

Row 2: As Operations we use (in this order) $(2) := -1\cdot(2)$, $(2) := (2)-5(3)$, $(4) := (4)-2(3)$, the swap of column (2) and (3) and $(1) := (1) - 3(2)$. Now, we get the unimodular matrix

$$U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -3 & 1 & -5 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the Hermite normal form

$$\text{HNF}(A) = A'U_2 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Hence, the transformationmatrix U is

$$U = U_1 \cdot U_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 4 & -1 & 7 & 1 \\ -3 & 1 & -5 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The system of equations has an integer solution, iff

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} b = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} b = \begin{pmatrix} \frac{1}{2}b_1 \\ b_2 \end{pmatrix} \in \mathbb{Z}^2.$$

This is the case if b_1 is even and b_2 is arbitrary.

b) Let b_1,b_2 be arbitrary but fixed with b_1 even and $y = \begin{pmatrix} \frac{1}{2}b_1 \\ b_2 \\ 0 \\ 0 \end{pmatrix}$. Then it is

$$Uy = \begin{pmatrix} 0 \\ 2b_1 - b_2 \\ -\frac{3}{2}b_1 + b_2 \\ 0 \end{pmatrix} \text{ and } Ue_3 = \begin{pmatrix} -1 \\ 7 \\ -5 \\ 0 \end{pmatrix}, \quad Ue_4 = \begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \end{pmatrix}.$$

So we know that then

$$\{x \in \mathbb{Z}^4 : Ax = b\} = \begin{pmatrix} 0 \\ 2b_1 - b_2 \\ -\frac{3}{2}b_1 + b_2 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} -1 \\ 7 \\ -5 \\ 0 \end{pmatrix} + \lambda_4 \begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \end{pmatrix}$$

for arbitrary $\lambda_3, \lambda_4 \in \mathbb{Z}$.

Problem 3.3

Let $G = (V, E)$ a digraph with $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$ and S_G the corresponding incidence matrix, i. e. $S_G = (a_{ij}) \in \{-1, 0, 1\}^{n \times m}$, s. t.

$$a_{ij} = \begin{cases} -1 & \text{if } e_j = (v_i, v_k) \text{ for some } k \in [n], \\ 1 & \text{if } e_j = (v_k, v_i) \text{ for some } k \in [n], \\ 0 & \text{else.} \end{cases}$$

Since $\operatorname{rank}(S_G) \le n - 1$ (why?), we remove one row from S_G (w. l. o. g.[1], the first) $\hat{S}_G$.

a) Show: an $(n-1) \times (n-1)$ submatrix of $\hat{S}_G$ is regular if and only if the corresponding subgraph of G (not considering the direction of the edges) is a spanning tree of G.

b) Let G be a tree and $x^i \in \mathbb{R}^m$, defined by

$$x^i_j = \begin{cases} -1 & \text{if the edge } e_j \text{ belongs to the path between } v_1 \text{ and } v_i \text{ within the tree} \\ & \text{and } e_j \text{ points in direction of } v_1, \\ 1 & \text{if the edge } e_j \text{ belongs to the path between } v_1 \text{ and } v_i \text{ within the tree} \\ & \text{and } e_j \text{ points in direction of } v_i, \\ 0 & \text{else.} \end{cases}$$

Show: $S_G x^i = -u^1 + u^i$ (where u^k denotes the k-th unit vector). What does x^i encode?

c) If G is a tree then $(\hat{S}_G^{-1})_{i-1} = x^i$.

d) If $P = \{x \in \mathbb{R}^m : \hat{S}_G x = u^{n-1}, x \ge 0\}$, then $P_I = P$ and every vertex of P corresponds to exactly one (directed) v_1, v_n-path within G.

e) In general, P is unbounded.

Solution to problem 3.3

a) Obviously the columns of $\hat{S}_G$ belonging to an (undirected) cycle are linearly dependent. If there are no cycles, the $n-1$ edges must form an (undirected) spanning tree. In this case it is easy to see from expanding the determinant along rows corresponding to leafs of (sub-)trees (every tree has leafs!) we obtain $|\det(\hat{S}_G)| = 1$, inductively.

[1] without loss of generality

b) Consider the (unique) v_1, v_i-path W within the tree, aligned by the „-1"-entries in x^i. Since there are exactly two edges incident with every vertex within W, it follows $S_G x^i = -u^1 + u^i$. This fact can be interpreted as the flow conservation of the one unit flow W from $s = v_1$ to $t = v_i$, therefore implying that the sum of incomming and outgoing flows in any inner vertex of the path equal zero.

c) Let $\hat{u}^k = u^{k-1} \in \mathbb{R}^{n-1}$. By deleting the first row of $S_G x^i = -u^1 + u^i$ we obtain $\hat{S}_G x^i = \hat{u}^i = u^{i-1}$. Hence $x^i = \hat{S}_G^{-1} u^{i-1}$ is the $(i-1)$-th column of $\hat{S}_G^{-1}$

d) From (a,b,c) it follows that the basic feasible solutions $x_B^* = (\hat{S}_G)_B^{-1} u^{n-1}$ (the vertices of P) are the (undirected) spanning trees in G (all other edges belong to no basis variables and get value 0). Since $x_B^* \geq 0$ it follows that the $(n-1)$-th column of $(S_G)_B^{-1}$ contains no negativ entry, i.e. the v_1, v_n-path in this tree has to be directed. Further: All arcs e_i on the v_1, v_n-path get the value $y_i = 1$ and all other arcs of the tree get the value $y_i = 0$. Hence, every vertex of P is integer and there is a one-to-one correspondence between this vertices and the v_1, v_n-pathes.

(Every basis corresponds with a spanning tree and every vertex corresponds with a v_1, v_n-path. Since many spanning trees can include the same v_1, v_n-path, the basis behind a basis solution does not need to be unique!)

e) Let W be a (shortest) v_1, v_n-path and C a cycle with no edge in common with W. Set $y_i = 1$ for all $e_i \in W$ and $y_i = N$ for all $e_i \in C$ ($N \in \mathbb{N}$ arbitrary). Futthermore, let $y_i = 0$ for all other edges. Then y is feasible and it follows the claim. (P is the Minkowski sum of the convex hull of directed v_1, v_n-pathes and the cone of all directed simple cycles)