Discrete Optimization (MA 3502), WiSe 2012/13

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Problem Sheet 3

Problem 3.1
Determine

a) all integer solutions of the equation $30 \cdot x_1 + 42 \cdot x_2 + 105 \cdot x_3 = 48$.

b) all $\alpha \in \mathbb{Z}$ such that the Linear Diophantine Equation $30 \cdot x_1 + 42 \cdot x_2 + 105 \cdot x_3 = \alpha$ is solvable, but for all possible solutions $x_i \neq 1$, $i \in \{1, 2, 3\}$, holds.

Problem 3.2
Consider the linear system of equations

$$Ax = b, \quad A = \begin{pmatrix} 4 & 2 & 2 & 2 \\ 1 & 3 & 4 & 5 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{Z}^2.$$

Determine all vectors b, such that there exists an integer solution x of the system and for each such b the set $\{x \in \mathbb{Z}^4 : Ax = b\}$.

Please turn over!

Problem 3.3

Let $G = (V, E)$ a digraph with $V = \{v_1, \dots, v_n\}$ and $E = \{e_1, \dots, e_m\}$ and S_G the corresponding incidence matrix, i. e. $S_G = (a_{ij}) \in \{-1, 0, 1\}^{n \times m}$, s. t.

$$a_{ij} = \begin{cases} -1 & \text{if } e_j = (v_i, v_k) \text{ for some } k \in [n], \\ 1 & \text{if } e_j = (v_k, v_i) \text{ for some } k \in [n], \\ 0 & \text{else.} \end{cases}$$

Since $\operatorname{rank}(S_G) \leq n - 1$ (why?), we remove one row from S_G (w. l. o. g.[1], the first) $\hat{S}_G$.

a) Show: an $(n-1) \times (n-1)$ submatrix of $\hat{S}_G$ is regular if and only if the corresponding subgraph of G (not considering the direction of the edges) is a spanning tree of G.

b) Let G be a tree and $x^i \in \mathbb{R}^m$, defined by

$$x^i_j = \begin{cases} -1 & \text{if the edge } e_j \text{ belongs to the path between } v_1 \text{ and } v_i \text{ within the tree} \\ & \text{and } e_j \text{ points in direction of } v_1, \\ 1 & \text{if the edge } e_j \text{ belongs to the path between } v_1 \text{ and } v_i \text{ within the tree} \\ & \text{and } e_j \text{ points in direction of } v_i, \\ 0 & \text{else.} \end{cases}$$

Show: $S_G x^i = -u_1 + u_i$ (where u_k denotes the k-th unit vector). What does x^i encode?

c) If G is a tree then $(\hat{S}_G^{-1})_{i-1} = x^i$.

d) If $P = \{x \in \mathbb{R}^m : \hat{S}_G x = u_{n-1}, x \geq 0\}$, then $P_I = P$ and every vertex of P corresponds to exactly one (directed) v_1, v_n-path within G.

e) In general, P is unbounded.

Submission: by Monday, 14:00 in the week following the exercise class, in the metal cupboard at the beginning of the M9-finger or in my office.

Please submit your work in groups.

[1] without loss of generality