Discrete Optimization (MA 3502), WiSe 2012/13

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Problem Sheet 4

Problem 4.1
Show that any matrix $A \in \{-1, 0, 1\}^{m\times n}$ is totally unimodular, iff the polytope

$$P = \{x \in \mathbb{R}^n : a \leq Ax \leq b,\ c \leq x \leq d\}$$

is integral for all $a, b \in \mathbb{Z}^m$ and $c, d \in \mathbb{Z}^n$.

Solution to problem 4.1

The system $a \leq Ax \leq b$, $c \leq x \leq d$ may be represented as

$$\begin{array}{rcl} Ax & \leq & b \\ -Ax & \leq & -a \\ -I_n x & \leq & -c \\ I_n x & \leq & d \end{array}.$$

Thus the statement follows from Corollary 3.2.13 and Remark 3.3.1.

Problem 4.2

a) For the following matrices, determine whether they are unimodular or even totally unimodular. If the matrices are not unimodular (respectively totally unimodular), identify appropriate submatrices whose determinant is not in $\{-1, 0, 1\}$.

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1 & 0 \\ -1 & -1 \\ -1 & 1 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

b) Describe the idea of a (polynomial) algorithm which tests whether a given matrix A fulfills the conditions in Corollary 3.3.3.

Solution to problem 4.2

a) (i) A_1 is unimodular (it has determinant 1), but not totally unimodular since $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is a square submatrix with determinant -2.

(ii) A_2 is not unimodular since $\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$ is a square submatrix of maximal size with determinant -2. For the same reason, A_2 is also not totally unimodular.

(iii) A_3 is unimodular and totally unimodular.

b) Surely, it is quite easy to check, if all columns have at most two non-zero entries. Thus assuming this to be true, we construct a graph G which is is bipartite iff the partition as described in Corollary 2.24 is possible. We assign one vertex to each row of A. If a column of A contains two 1 entries or two -1 entries we connect the vertices of the corresponding rows with an edge. If a column contains one 1 and one -1 entry we add an additional vertex to the graph and connect this new vertex with the vertices corresponding to the -1 and 1 entries.
Obviously, every partition of the rows of A (as in 2.24) leads to a partition of the vertices in the constructed graph which shows that it is bipartite and vice versa.
As we know it is possible to check if a graph is bipartite in polynomial time.

Problem 4.3

a) Which of the following statements are true and which are false? Give a proof or a counter-example.

(i) If A is a square, totally unimodular matrix, then all eigenvalues of A are in $\{-1, 0, 1\}$

(ii) If $A \in \mathbb{Z}^{d\times n}$ is totally unimodular and $B \in \{-1, 0, 1\}^{m\times n}$, then $\begin{pmatrix} A \\ B \end{pmatrix}$ is unimodular.

(iii) If $A \in \mathbb{Z}^{n\times n}$ is totally unimodular and has rank n, then $\mathrm{HNF}(A) = I$ (where I denotes the identy matrix).

(iv) If A, B are totally unimodular, then $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is also totally unimodular.

b) Show that every totally unimodular matrix A has the following properties:

(i) Every regular submatrix of A has at least one row with an odd number of non-zero entries.

(ii) The sum over all entries in every square submatrix of A with even row and column sums is divisble by 4.

Remark: Both properties characterise total unimodularity!

Solution to problem 4.3

a) (i) False, e.g. $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is totally unimodular and 2 is an eigenvalue.

(ii) False, e.g. $A = (1, 1)$ is totally unimodular and $B = (1, -1) \in \{-1, 0, 1\}^{1\times 2}$, but $\begin{pmatrix} A \\ B \end{pmatrix}$ has determinant 2, so is not unimodular.

(iii) True: The determinant of $\mathrm{HNF}(A)$ is the same as that of A up to sign. Since A has rank n and is totally unimodular, it has determinant ± 1. Thus $\mathrm{HNF}(A)$ has determinant 1 (all entries on the diagonal must be positive, and the determinant is their product). Furthermore, since $\mathrm{HNF}(A)$ is integral, the values on the diagonal are all 1, and therefore the values elsewhere must be 0. Thus $\mathrm{HNF}(A) = I$.

(iv) True: $P = \{\begin{pmatrix} x \\ y \end{pmatrix} : \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ c \end{pmatrix}, x, y \geq 0\}$ is for any right-hand side $\begin{pmatrix} b \\ c \end{pmatrix}$ the carthesian product $P_1 \times P_2$ of the integral polytopes $P_1 = \{x : Ax = b, x \geq 0\}$ and $P_2 = \{y : By = c, y \geq 0\}$ and therefore again integral. Hence $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ must be totally unimodular.

Alternatively: Any square submatrix C has the form $C = \begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix}$, where A' and B' are submatrices of A and B respectively. If these are not square, then C has determinant 0. If they are square, then $\det(C) = \det(A)\det(B) \in \{-1, 0, 1\}$.

b) By Theorem 2.23 it follows from totally unimodularity of A that for each $k \times k$-submatrix B of A there exists a partition (I, J) of the columns b_i of B such that $\sum_{i \in I} b_i - \sum_{i \in J} b_i \in \{0, \pm 1\}^k$, in other words there exists an $x \in \{-1, +1\}^k$ such that $Bx = \{0, \pm 1\}^k$.

   (i) We assume that every row of B contains an even number of non-zero entries. This implies Bx even and with $Bx \in \{-1, 0, 1\}^k$ we obtain $Bx = 0$ and so B singular.

   (ii) Using (i) it follows that the partition of the columns of B into submatrices B_1 and B_2 such that the sum of the columns of B_1 yields the same vector as the sum of the columns of B_2. Let S be the sum of all entries of B and let S_i be the sum of all entries of B_i, $i = 1, 2$. So we have $S_1 = S_2$ and S_1 even (as all column sums are even). Thus it follows that $S = S_1 + S_2 = 2S_1$ is divisible by 4.

Please note that with the assumption from above one can prove the equivalence of all the statements.

Problem 4.4

Let $G = (V, E)$ be a connected graph, $|V| = n$, $|E| = m$ and S_G the incidence matrix of G. Show that:

a) The set $\{x \in \{0, 1\}^m : S_G x = \mathbb{1}\}$ describes exactly the set of perfect matchings in G.

b) All vertices x^* of the polytope $P = \{x \in \mathbb{R}^m : S_G x = \mathbb{1},\ x \geq 0\}$ fulfill $2x^* \in \mathbb{Z}^m$ and (moreover) $x^* \in \mathbb{Z}^m$ if G is bipartite (i.e. $P = P_I$ and the vertices of P are exactly the incidence vectors of the perfect matchings in G).

Solution to problem 4.4

a) With $x_j = 1$ if edge j is in the matching and $x_j = 0$ if not $S_G x = \mathbb{1}$ encodes that each vertex belongs to exacty 1 edge.

b) first option:
First note: If G is bipartite, then S_G is totally unimodular (Theorem 3.4.1) and therefore $x^* \in \mathbb{Z}^m$ for all vertices $x*$ of P.

Now we prove: For general graphs G, all vertices x^* of the polytope $P = \{x \in \mathbb{R}^m : S_G x = \mathbb{1},\ x \geq 0\}$ satisfy $2x^* \in \mathbb{Z}^m$.
Consider the graph $G'' = (V'', E'')$ with two vertices $v_{i1}, v_{i2} \in V''$ for each vertex $v_i \in V$ and edges $v_{i1}v_{j2}, v_{i2}v_{j1} \in E''$ for each edge $v_i v_j \in E$.
Obviously G'' is bipartite and so all vertices y^* of the corresponding relaxed matching-polytope

$Q = \{y \in \mathbb{R}^{2m} : S_{G''}y = \mathbb{1},\ y \geq 0\}$ are integral. Now one can easily check from correspondence of bases between P and Q that the vertices of P are exactly the vectors x with $x_i := \begin{cases} 0 & \text{, if } y_i = y_{i+m} = 0 \\ 1 & \text{, if } y_i = y_{i+m} = 1 \\ \frac{1}{2} & \text{, else} \end{cases}$ and y vertex of the polytope belonging to G'' and so every vertex $x \in P$ is $\in \left\{0, \frac{1}{2}, 1\right\}^m$.

second option:

Let x^* be a vertex of P and let B_{x^*} be a basis matrix of x^* which is invertible (notice that the basis matrix is not unique because x^* is degenerate in general).

W.l.o.g. the basis matrix has the structure $B_{x^*} = \begin{pmatrix} A_1 & 0 & & 0 \\ 0 & A_2 & & \\ & 0 & \ddots & 0 \\ & & 0 & A_k \end{pmatrix}$ with $A_1, \ldots, A_k$ corresponding to the connected components. By Cramer's rule we obtain $x_i^* = \frac{\det B_{x^*}^i}{\det B_{x^*}}$.

To simplify we consider $i = 1$ (the other cases follow analogously).

Thus $B_{x^*}^1 = \begin{pmatrix} \mathbb{1} & A_1' & 0 & & 0 \\ \mathbb{1} & 0 & A_2 & & \\ \vdots & & 0 & \ddots & 0 \\ \mathbb{1} & & & 0 & A_k \end{pmatrix}$ and therefore $x_1^* = \frac{\det(\tilde{B_{x^*}^1})}{\det(B_{x^*})} = \frac{\det\left(\mathbb{1} A_1'\right) \cdot \prod_{i=2}^k \det\left(A_i\right)}{\prod_{i=1}^k \det A_i} = \frac{\det\left(\mathbb{1} A_1'\right)}{\det A_1}$ with $\det A_1 \in \{-2, -1, 1, 2\}$. And for G bipartit it is $\det A_1 \in \{-1, 1\}$.