Discrete Optimization (MA 3502), WiSe 2012/13

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Problem Sheet 4

Problem 4.1
Show that any matrix $A \in \{-1,0,1\}^{m\times n}$ is totally unimodular, iff the polytope

$$P = \{x \in \mathbb{R}^n : a \le Ax \le b,\ c \le x \le d\}$$

is integral for all $a, b \in \mathbb{Z}^m$ and $c, d \in \mathbb{Z}^n$.

Problem 4.2

a) For the following matrices, determine whether they are unimodular or even totally unimodular. If the matrices are not unimodular (respectively totally unimodular), identify appropriate submatrices whose determinant is not in $\{-1,0,1\}$.

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1 & 0 \\ -1 & -1 \\ -1 & 1 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

b) Describe the idea of a (polynomial) algorithm which tests whether a given matrix A fulfills the conditions in Corollary 3.3.3.

Problem 4.3

a) Which of the following statements are true and which are false? Give a proof or a counter-example.

(i) If A is a square, totally unimodular matrix, then all eigenvalues of A are in $\{-1,0,1\}$

(ii) If $A \in \mathbb{Z}^{d\times n}$ is totally unimodular and $B \in \{-1,0,1\}^{m\times n}$, then $\begin{pmatrix} A \\ B \end{pmatrix}$ is unimodular.

(iii) If $A \in \mathbb{Z}^{n\times n}$ is totally unimodular and has rank n, then $\mathrm{HNF}(A) = I$ (where I denotes the identy matrix).

(iv) If A, B are totally unimodular, then $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is also totally unimodular.

b) Show that every totally unimodular matrix A has the following properties:

(i) Every regular submatrix of A has at least one row with an odd number of non-zero entries.

(ii) The sum over all entries in every square submatrix of A with even row and column sums is divisble by 4.

Remark: Both properties characterise total unimodularity!

Please turn over!

Problem 4.4
Let $G = (V, E)$ be a connected graph, $|V| = n$, $|E| = m$ and S_G the incidence matrix of G. Show that:

a) The set $\{x \in \{0,1\}^m : S_G x = \mathbb{1}\}$ describes exactly the set of perfect matchings in G.

b) All vertices x^* of the polytope $P = \{x \in \mathbb{R}^m : S_G x = \mathbb{1},\ x \geq 0\}$ fulfill $2x^* \in \mathbb{Z}^m$ and (moreover) $x^* \in \mathbb{Z}^m$ if G is bipartite (i.e. $P = P_I$ and the vertices of P are exactly the incidence vectors of the perfect matchings in G).

Submission: by Monday, 14:00 in the week following the exercise class, in the metal cupboard at the beginning of the M9-finger or in my office.

Please submit your work in groups.