Discrete Optimization (MA 3502), WiSe 2012/13

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Problem Sheet 5

Problem 5.1
Let

$$C := \text{pos}\left\{ \begin{pmatrix} 4/3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 4/3 \end{pmatrix} \right\}.$$

a) Guess an inclusion-minimal Hilbert basis H for C and prove that your H actually is a Hilbert basis.

b) Prove that your H is a unique inclusion-minimal Hilbert basis.

Problem 5.2
Let v^1, v^2 be two vectors in $\mathbb{R}^2$, $P = \text{pos}\{v^1, v^2\}$, and H the minimal Hilbert basis of P.

a) Give values for v^1 and v^2, s. t. $|H| = 2$. Determine H.

b) Give values for v^1 and v^2, s. t. $|H| = 5$ and all elements of H are vertices of $\text{conv}(P \cap \mathbb{Z}^2 \setminus \{0\})$. Determine H.

c) Show that for any $n \in \mathbb{N}$ there are rational vectors v^1 and v^2, s. t. $|H| \geq n$.

Problem 5.3
Let $v^1, \dots, v^k \in \mathbb{Z}^n$ and suppose $K := \text{pos}\{v^1, \dots, v^k\}$ is a pointed cone. A *zonotope* Z is a Minkowski-sum of line segments, $Z = \sum_{i=1}^k [\alpha_i, \beta_i] z^i$ for some $\alpha_i < \beta_i$ and $z^i \in \mathbb{R}^n$. A *parallelotope* is a zonotope $\sum_{i=1}^k [\alpha_i, \beta_i] z^i$, where all z^i are linearly independent.

Show: the minimal Hilbert basis R of K is a subset of the integer points within

a) the zonotope
$$\sum_{i=1}^k [0,1] v^i = \left\{ \sum_{i=1}^k \lambda_i v^i : \lambda_i \in [0,1] \right\}.$$

b) the union of parallelotopes
$$\bigcup_{B \in \mathcal{B}} \sum_{i \in \mathcal{B}} [0,1] v^i,$$
where $\mathcal{B} := \{B \subset [k] : \{v_i : i \in B\} \text{ is a basis of } \mathbb{R}^n\}$

Please turn over!

Problem 5.4

Let $G = (V, E)$ a graph and S_G the corresponding incidence matrix. Consider again the perfect-matching polytope $P = \{x \in \mathbb{R}^m : S_G x = \mathbb{1},\ x \geq 0\}$ as given in Problem 4.4.

a) Give an example of a graph G such that P has a fractional vertex.

b) Let $C = (U, E_U)$ denote an odd cycle in G with nodes $U = \{v_1, \ldots, v_{2n+1}\}$ and edges $E_U = \{\{v_i, v_{i+1}\} : i \in [2n]\} \cup \{\{v_1, v_{2n+1}\}\} \subset E$. Show:
Every incidence vector x of a (perfect) matching, fulfills additionaly to $S_G x = \mathbb{1},\ x \geq 0$ the condition

$$\sum_{e \in E_U} x_e \leq \frac{|U| - 1}{2}.$$

c) Suppose the induced subgraph $G(U)$ of G contains more edges than E_U. Show that the inequality $\sum_{e \in E_U} x_e \leq \frac{|U|-1}{2}$ is redundant (i.e. it can be obtained from adding other inequalities in $S_G x = \mathbb{1}$, $x \geq 0$, or inequalities belonging to other odd cycles).

Submission: by Monday, 14:00 in the week following the exercise class, in the metal cupboard at the beginning of the M9-finger or in my office.

Please submit your work in groups.