Technische Universität München
Zentrum Mathematik

Discrete Optimization (MA 3502), WiSe 2012/13

Prof. Dr. P. Gritzmann, Dr. René Brandenberg

Problem Sheet 6

Problem 6.1
Let
$$P = \left\{ x \in \mathbb{R}^2 : \begin{pmatrix} -1 & 0 \\ 1 & 2 \\ 1 & -2 \end{pmatrix} x \leq \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\} = \operatorname{conv} \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \right\}.$$

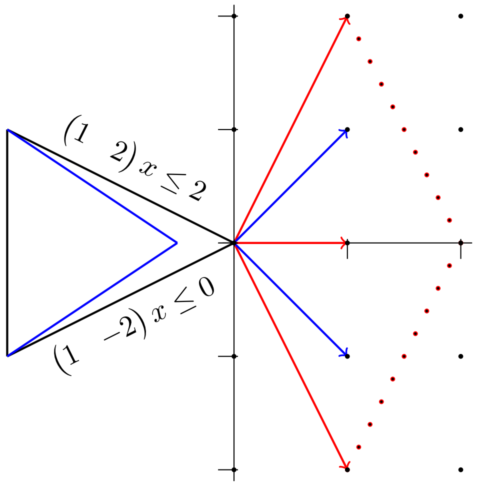
a) Show that the Chvátal-Gomory closure (Chvátal-Gomory differential) of P is
$$P_1 = \operatorname{conv}\{(0,0)^T, (0,1)^T, (1/2, 1/2)^T\}.$$

b) Show that the Chvátal-Gomory closure of P_1 is $P_2 = P_I = \operatorname{conv}\{(0,0)^T, (0,1)^T\}$.

c) Let $k \in \mathbb{N}$ and
$$Q = \operatorname{conv}\{(0,0)^T, (0,1)^T, (k, 1/2)^T\}.$$
Show: $Q_{2k-1} \neq Q_I$ and $Q_{2k} = Q_I (= P_I)$ by proving $Q_i = \operatorname{conv}\{(0,0)^T, (0,1)^T, (k - i/2, 1/2)\}$, where Q_i denotes the i-th Chvatal-Gomory closure.
What properties does the cut $x_1 \leq 0$ in connection with Q have? (e. g., validity, R-Cut, Gomory-Cut?)

Solution to problem 6.1

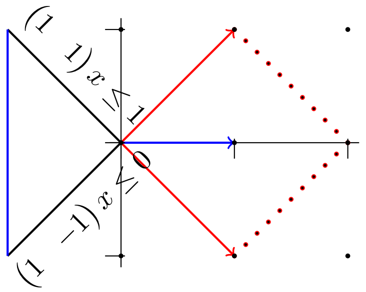
a) It suffices to construct the Hilbert basis of the cone of the normal vectors of the unique fractional vertex and to add the corresponding (irredundant) cuts. The Hilbert basis is
$$\left\{ (1,2)^T, (1,1)^T, (1,0)^T, (1,-1)^T, (1,-2)^T \right\}.$$
Thus the vectors $(1,1)^T$ and $(1,-1)^T$ yield irredundant cuts. The claim follows.



b) Again it suffices to construct the Hilbert basis of the cone of the normal vectors of the unique fractional vertex and to add the corresponding (irredundant) cuts. The Hilbert basis is

$$\left\{(1,1)^T, (1,0)^T, (1,-1)^T\right\}.$$

The vector $(1,0)^T$ yields the desired cut to obtain $P_2 = P_I$.



c) We show by induction over i that

$$Q_{2k-i} = Q^i := \mathrm{conv}\{(0,0)^T, (0,1)^T, (i/2, 1/2)^T\}.$$

It was already shown in (a,b) that $Q^0 = (Q^1)_1$ and $Q^1 = (Q^2)_1$. It remains to show that $Q^{i-1} = (Q^i)_1$ for $3 \le i \le 2k$. We can describe Q^i by the following inequlities: $x_1 + ix_2 \le i$, $x_1 - ix_2 \le 0$ and $-x_1 \le 0$, from which the first two are the only ones which are active in the unique fractional vertex $(i/2, 1/2)^T$. The corresponding Hilbert basis is $H = \{(1,j)^T : -i \le j \le i\}$ and the respectively related R-Cut is $x_1 + jx_2 \le \lfloor (i+j)/2 \rfloor$, i. e. only if $i+j$ is odd we obtain proper cuts which are $x_1 + jx_2 \le (i+j-1)/2$. From inserting one may verify $((i-1)/2, 1/2)^T$ fulfills all conditions and therefore we have $(Q^i)_1 \supset Q^{i-1}$. For $j = i-1$ or $j = -i+1$ we obtain the constraints which are active in $((i-1)/2, 1/2)^T$ from Q^{i-1} as R-Cuts to Q^i. Hence $(Q^i)_1 \subset Q^{i-1}$ and altogether $(Q^i)_1 = Q^{i-1}$.

Problem 6.2

The gomory cutting plane algorithm computes a soultion of an ILP as follows:

Let P_0 the feasible set of an LP-relaxation of the ILP. Iterate over i: Compute the optimal solution x^i over P_i and check if it is integral. If yes, we are done. Otherwise add a gomory cut to P_i

(i) with respect to objective c, if $c^T x^i$ is fractional

(ii) with respect to u^k, $k \in [n]$, if $c^T x^i$ and x^j, $j \le k-1$ are integral, but x^k is fractional.

and call the new polyhedron P_{i+1}.

Use the Gomory cutting plane algorithm to compute an optimal solution of the ILP

$$\begin{array}{rrcrcl} \max & 5x_1 & + & 6x_2 & & \\ & 3x_1 & + & 5x_2 & \le & 15 \\ & 3x_1 & - & 5x_2 & \le & 0 \\ & & & x & \in & \mathbb{Z}^2 \end{array}$$

Draw a sketch of the situation in each step, get the active constraints in the current basic solution from the sketch, and compute the exact solution from the corresponding equation system.

Solution to problem 6.2

It applies $x^1 = (5/2, 3/2)^T$ and the corresponding basis is $B_1 = \{1, 2\}$, the basis matrix

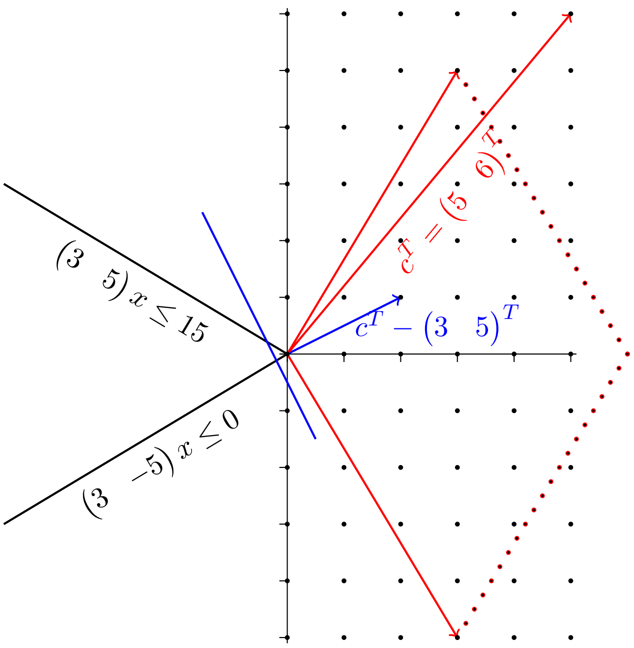
$$A_{B_1} = \begin{pmatrix} 3 & 5 \\ 3 & -5 \end{pmatrix} \text{ und } A_{B_1}^{-1} = 1/30 \begin{pmatrix} 5 & 5 \\ 3 & -3 \end{pmatrix}.$$

Since $c^T x^1 = 43/2$ is fractional we determine the Gomory-cut to the objective:

$$\left(c^T - \left\lfloor c^T A_{B_1}^{-1} \right\rfloor A_{B_1}\right) x \leq \lfloor c^T x^1 \rfloor - \left\lfloor c^T A_{B_1}^{-1} \right\rfloor A_{B_1} x^1$$

$$\left(\begin{pmatrix} 5 & 6 \end{pmatrix} - \underbrace{\left\lfloor \frac{1}{30} \begin{pmatrix} 5 & 6 \end{pmatrix} \begin{pmatrix} 5 & 5 \\ 3 & -3 \end{pmatrix} \right\rfloor}_{\begin{pmatrix} 1 & 0 \end{pmatrix}} \begin{pmatrix} 3 & 5 \\ 3 & -5 \end{pmatrix} \right) x \leq \underbrace{\left\lfloor \begin{pmatrix} 5 & 6 \end{pmatrix} \begin{pmatrix} 5/2 \\ 3/2 \end{pmatrix} \right\rfloor}_{21} - \underbrace{\left\lfloor \frac{1}{30} \begin{pmatrix} 5 & 6 \end{pmatrix} \begin{pmatrix} 5 & 5 \\ 3 & -3 \end{pmatrix} \right\rfloor}_{\begin{pmatrix} 1 & 0 \end{pmatrix}} \begin{pmatrix} 3 & 5 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} 5/2 \\ 3/2 \end{pmatrix}$$

i. e. the Gomory-cut result in $2x_1 + x_2 \leq 6$, which we add as new last inequality to our ILP.



Obviously $B_2 = \{1, 3\}$ and it follows $x^2 = (15/7, 12/7)^T$. Further

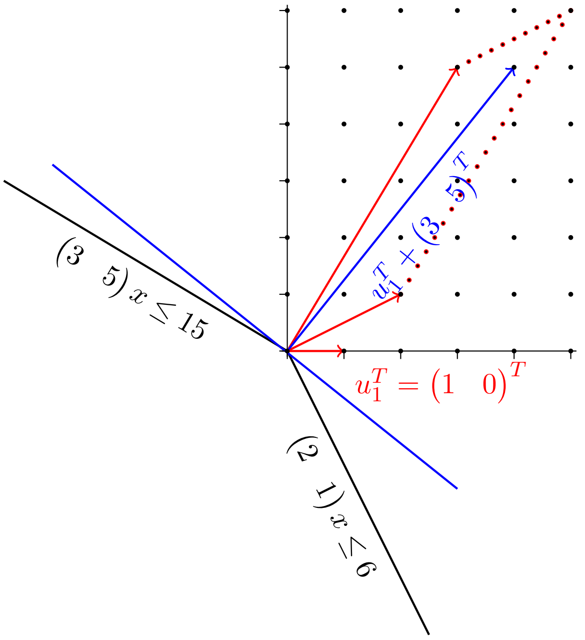
$$A_{B_2} = \begin{pmatrix} 3 & 5 \\ 2 & 1 \end{pmatrix} \text{ and } A_{B_2}^{-1} = 1/7 \begin{pmatrix} -1 & 5 \\ 2 & -3 \end{pmatrix}.$$

It applies $c^T x^2 = 21$, i. e. we determine the Gomory-cut to the first component:

$$\left(u_1^T - \left\lfloor u_1^T A_{B_2}^{-1} \right\rfloor A_{B_2}\right) x \leq \lfloor u_1^T x^2 \rfloor - \left\lfloor u_1^T A_{B_2}^{-1} \right\rfloor A_{B_2} x^2$$

$$\left(\begin{pmatrix}1 & 0\end{pmatrix} - \underbrace{\left\lfloor \frac{1}{7}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}-1 & 5\\ 2 & -3\end{pmatrix}\right\rfloor}_{\begin{pmatrix}-1 & 0\end{pmatrix}} \begin{pmatrix}3 & 5\\ 2 & 1\end{pmatrix}\right) x \le \underbrace{\left\lfloor \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}15/7\\ 12/7\end{pmatrix}\right\rfloor}_{2} - \underbrace{\left\lfloor \frac{1}{7}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}-1 & 5\\ 2 & -3\end{pmatrix}\right\rfloor}_{\begin{pmatrix}-1 & 0\end{pmatrix}} \begin{pmatrix}3 & 5\\ 2 & 1\end{pmatrix}\begin{pmatrix}15/7\\ 12/7\end{pmatrix}$$

The second Gomory-cut is therewith $4x_1 + 5x_2 \le 17$, which we add again as the last inequality to our ILP.



A short compare of the objective values of the basic solutions to the basis $\{1,4\}$ or $\{3,4\}$ shows that $B_3 = \{3,4\}$ and $x^3 = (13/6, 5/3)^T$. It applies

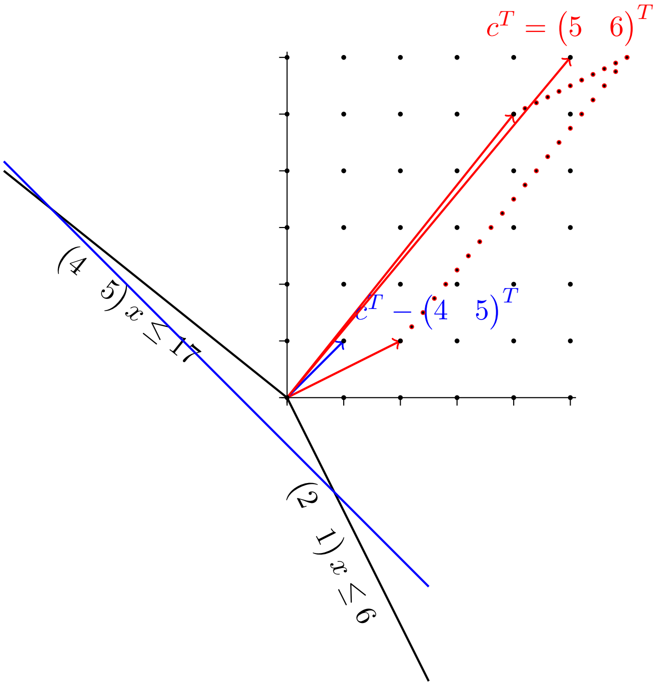
$$A_{B_3} = \begin{pmatrix}2 & 1\\ 4 & 5\end{pmatrix} \text{ und } A_{B_3}^{-1} = 1/6 \begin{pmatrix}5 & -1\\ -4 & 2\end{pmatrix}.$$

It applies $c^T x^3 = 125/6$, i. e. we determine the Gomory-cut to the objective:

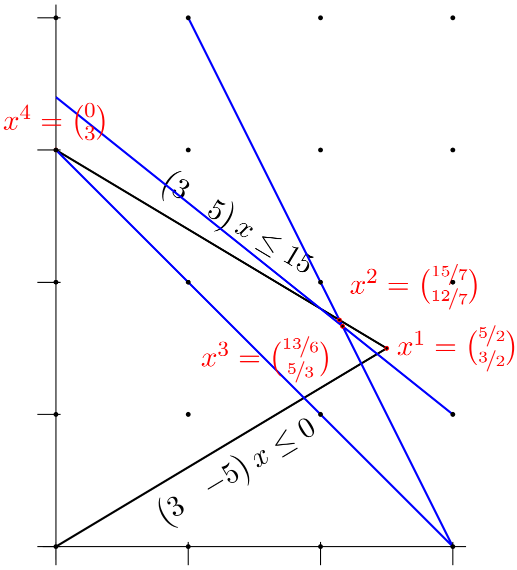
$$\left(c^T - \left\lfloor c^T A_{B_3}^{-1}\right\rfloor A_{B_3}\right) x \le \left\lfloor c^T x^3\right\rfloor - \left\lfloor c^T A_{B_3}^{-1}\right\rfloor A_{B_3} x^3$$

$$\left(\begin{pmatrix}5 & 6\end{pmatrix} - \underbrace{\left\lfloor \frac{1}{6}\begin{pmatrix}5 & 6\end{pmatrix}\begin{pmatrix}5 & -1\\ -4 & 2\end{pmatrix}\right\rfloor}_{\begin{pmatrix}0 & 1\end{pmatrix}} \begin{pmatrix}2 & 1\\ 4 & 5\end{pmatrix}\right) x \le \underbrace{\left\lfloor \begin{pmatrix}5 & 6\end{pmatrix}\begin{pmatrix}13/6\\ 10/6\end{pmatrix}\right\rfloor}_{20} - \underbrace{\left\lfloor \frac{1}{6}\begin{pmatrix}5 & 6\end{pmatrix}\begin{pmatrix}5 & -1\\ -4 & 2\end{pmatrix}\right\rfloor}_{\begin{pmatrix}0 & 1\end{pmatrix}} \begin{pmatrix}2 & 1\\ 4 & 5\end{pmatrix}\begin{pmatrix}13/6\\ 10/6\end{pmatrix}$$

The third Gomory-cut is $x_1 + x_2 \le 3$, which we add again as the last inequality to our ILP.

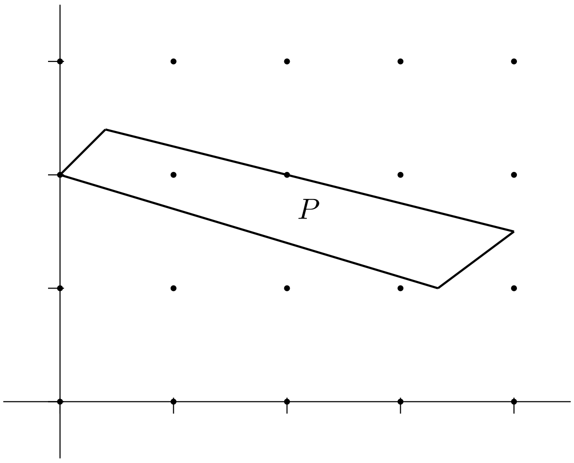


Since the basic solutions $(3,0)^T$ or $(-2,5)^T$ to the basis $\{3,5\}$ or $\{4,5\}$ are infeasible (they violate constraint 2 and 1, respectively) it is $B_4 = \{1,5\}$ the new optimal basis and $x^4 = (0,3)^T$ the new optimal solution of the LP-relaxation. Since x^4 is integral, it is an optimal solution to the ILP.



Problem 6.3
Let

$$P = \left\{ x \in \mathbb{R}^2 : \begin{pmatrix} -1 & 1 \\ 1 & 4 \\ 3 & -4 \\ -3 & -10 \end{pmatrix} x \le \begin{pmatrix} 2 \\ 10 \\ 6 \\ -20 \end{pmatrix} \right\} = \operatorname{conv} \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2/5 \\ 12/5 \end{pmatrix}, \begin{pmatrix} 4 \\ 3/2 \end{pmatrix}, \begin{pmatrix} 10/3 \\ 1 \end{pmatrix} \right\}$$



a) Draw P_I within the sketch of P.

b) Decide if P has Chvatal-Gomory rank 1 (Proof!).

c) Construct the Gomory-cut at $x^1 = \begin{pmatrix} 4 \\ 3/2 \end{pmatrix}$ with respect to $v = (v_1, 2)^T$, choosing $v_1 \in \mathbb{Z}$ such that v belongs to the Hilbert basis of the normal cone at x^1.

d) Construct the Gomory-cut at $x^2 = \begin{pmatrix} 10/3 \\ 1 \end{pmatrix}$ with respect to the first component. Why is it possible to get a better cut with the same outer-normal direction?

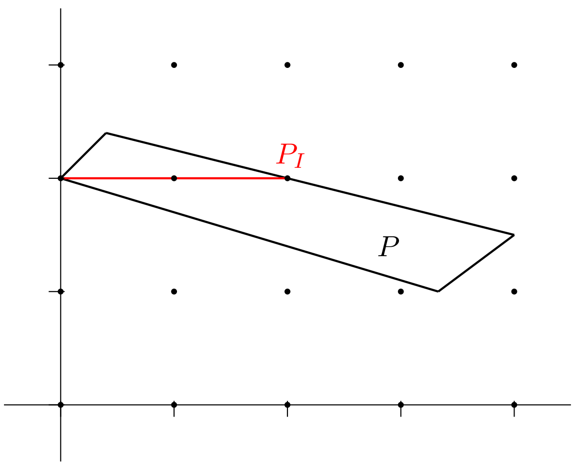
e) Let $Q = \{x \in \mathbb{R}^n : Ax \leq b\}$ a polyhedron, $(q^*)^T x \leq \lfloor (q^*)^T x^* \rfloor$ an R-cut, and $\alpha = \gcd(q_1^*, \ldots, q_n^*)$ the greatest common divisor of q^*'s components. Show:

$$\left(\frac{q^*}{\alpha}\right)^T x \leq \left\lfloor \frac{\lfloor (q^*)^T x^* \rfloor}{\alpha} \right\rfloor$$

is an R-cut with the same outer normal direction, which cuts at least as deep as the first cut, but possibly deeper (even if the first was a Gomory-cut).

Solution to problem 6.3

a)



b) To have $P' = P_I$, the missing inequalities $x_2 \leq 2$ and $x_2 \geq 2$ for the description of P_I must be R-cuts for P ($x_1 \in [0, 2]$ then implicitely follows from the valid inequalties for P, which stay valid for P_I).

$(0, 1)^T$ belongs to the normal cone of the vertex $(2/5, 12/5)^T$ and $\lfloor (0, 1)(2/5, 12/5)^T \rfloor = 2$. Hence $x_2 \leq 2$ is an R-cut.

$(0, -1)^T$ belongs to the normal cone of the vertex $(10/3, 1)^T$ and $\lfloor (0, -1)(10/3, 1)^T \rfloor = -1$. Thus $x_2 \geq 2$ is *not* an R-cut.

It follows $P' \neq P_I$ (which may be seen from the sketch in (a) as P touches the gridline $x_2 = 1$.

c) The normal cone at x^1 is $\text{pos}\{(1, 4)^T, (3, -4)^T\}$. Thus $(1, 2)^T = 5/8(1, 4)^T + 1/8(3, -4)^T$ belongs to the cone and therefore clearly to the Hilbert basis. Choosing $v_1 = 1$ we know that the Gomory-cut is $v^T x \leq \lfloor v^T x^1 \rfloor$, i. e. $x_1 + 2x_2 \leq 5$.

d) The gomory cut to the first component means $v = (1, 0)^T$, which is obviously not in the normal cone $\text{pos}\{(3, -4)^T, (-3, -10)^T\}$. The calculations for the gomory cut will result in the vector $q = (-2, -10)^T = 5/21(3, -4)^T + 19/21(-3, -10)^T$ within the Hilbertparallelotope. Thus the gomory cut is $-2x_1 - 10x_2 \leq \lfloor -20/3 - 10 \rfloor = -17$.

However, if we devide by two, we get $-x_1 - 5x_2 \leq -17/2$ and for all integer points in P it holds $-x_1 - 5x_2 \leq -9$ or $x_1 + 5x_2 \geq 9$.

e) (Observe, that this is just the generalization of the situation we discovered in (d)). First, if α divides the right-hand side $\lfloor (q^*)^T x^* \rfloor$ of the original R-cut, too, then the new cut and the old are equivalent. However, if that is not the case, we have

$$\left\lfloor \frac{\lfloor (q^*)^T x^* \rfloor}{\alpha} \right\rfloor < \frac{\lfloor (q^*)^T x^* \rfloor}{\alpha}$$

which means that the new cut is really deeper. It remains to show, that it is still an R-cut, which is a direct consequence of

$$\left\lfloor \frac{\lfloor (q^*)^T x^* \rfloor}{\alpha} \right\rfloor = \left\lfloor \left(\frac{q^*}{\alpha} \right)^T x^* \right\rfloor$$

(which still needs a short proof).