# Technische Universität München
Zentrum Mathematik

# Discrete Optimization (MA 3502), WiSe 2012/13

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## Problem Sheet 6

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**Problem 6.1**

Let

$$P = \left\{ x \in \mathbb{R}^2 : \begin{pmatrix} -1 & 0 \\ 1 & 2 \\ 1 & -2 \end{pmatrix} x \leq \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\} = \operatorname{conv} \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \right\}.$$

a) Show that the Chvátal-Gomory closure (Chvátal-Gomory differential) of $P$ is

$$P_1 = \operatorname{conv}\{(0,0)^T, (0,1)^T, (1/2, 1/2)^T\}.$$

b) Show that the Chvátal-Gomory closure of $P_1$ is $P_2 = P_I = \operatorname{conv}\{(0,0)^T, (0,1)^T\}$.

c) Let $k \in \mathbb{N}$ and

$$Q = \operatorname{conv}\{(0,0)^T, (0,1)^T, (k, 1/2)^T\}.$$

Show: $Q_{2k-1} \neq Q_I$ and $Q_{2k} = Q_I (= P_I)$ by proving $Q_i = \operatorname{conv}\{(0,0)^T, (0,1)^T, (k - i/2, 1/2)\}$, where $Q_i$ denotes the $i$-th Chvatal-Gomory closure.

What properties does the cut $x_1 \leq 0$ in connection with $Q$ have? (e. g., validity, R-Cut, Gomory-Cut?)

**Problem 6.2**

The gomory cutting plane algorithm computes a soultion of an ILP as follows:

Let $P_0$ the feasible set of an LP-relaxation of the ILP. Iterate over $i$: Compute the optimal solution $x^i$ over $P_i$ and check if it is integral. If yes, we are done. Otherwise add a gomory cut to $P_i$

(i) with respect to objective $c$, if $c^T x^i$ is fractional

(ii) with respect to $u^k$, $k \in [n]$, if $c^T x^i$ and $x^j$, $j \leq k-1$ are integral, but $x^k$ is fractional.

and call the new polyhedron $P_{i+1}$.

Use the Gomory cutting plane algorithm to compute an optimal solution of the ILP

$$\begin{array}{rrcrcr} \max & 5x_1 & + & 6x_2 & & \\ & 3x_1 & + & 5x_2 & \leq & 15 \\ & 3x_1 & - & 5x_2 & \leq & 0 \\ & & & x & \in & \mathbb{Z}^2 \end{array}$$

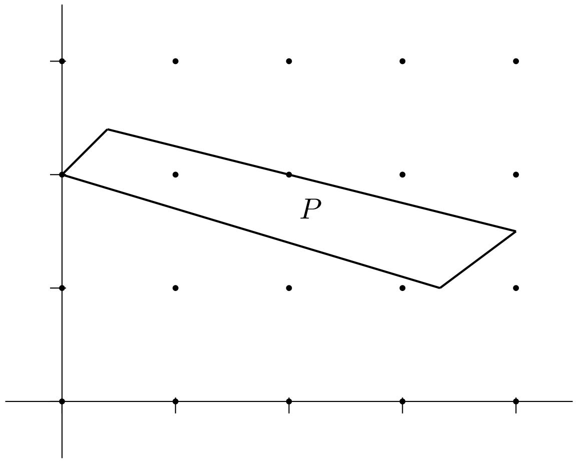
Draw a sketch of the situation in each step, get the active constraints in the current basic solution from the sketch, and compute the exact solution from the corresponding equation system.

**Please turn over!**

**Problem 6.3**

Let

$$P = \left\{ x \in \mathbb{R}^2 : \begin{pmatrix} -1 & 1 \\ 1 & 4 \\ 3 & -4 \\ -3 & -10 \end{pmatrix} x \leq \begin{pmatrix} 2 \\ 10 \\ 6 \\ -20 \end{pmatrix} \right\} = \text{conv} \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2/5 \\ 12/5 \end{pmatrix}, \begin{pmatrix} 4 \\ 3/2 \end{pmatrix}, \begin{pmatrix} 10/3 \\ 1 \end{pmatrix} \right\}$$




a) Draw $P_I$ within the sketch of $P$.

b) Decide if $P$ has Chvatal-Gomory rank 1 (Proof!).

c) Construct the Gomory-cut at $x^1 = \begin{pmatrix} 4 \\ 3/2 \end{pmatrix}$ with respect to $v = (v_1, 2)^T$, choosing $v_1 \in \mathbb{Z}$ such that $v$ belongs to the Hilbert basis of the normal cone at $x^1$.

d) Construct the Gomory-cut at $x^2 = \begin{pmatrix} 10/3 \\ 1 \end{pmatrix}$ with respect to the first component. Why is it possible to get a better cut with the same outer-normal direction?

e) Let $Q = \{x \in \mathbb{R}^n : Ax \leq b\}$ a polyhedron, $(q^*)^T x \leq \lfloor (q^*)^T x^* \rfloor$ an R-cut, and $\alpha = \gcd(q_1^*, \ldots, q_n^*)$ the greatest common divisor of $q^*$'s components. Show:

$$\left( \frac{q^*}{\alpha} \right)^T x \leq \left\lfloor \frac{\lfloor (q^*)^T x^* \rfloor}{\alpha} \right\rfloor$$

is an R-cut with the same outer normal direction, which cuts at least as deep as the first cut, but possibly deeper (even if the first was a Gomory-cut).

**Submission: by Monday, 14:00 in the week following the exercise class, in the metal cupboard at the beginning of the M9-finger or in my office.**

**Please submit your work in groups.**