Discrete Optimization (MA 3502), WiSe 2012/13

Prof. Dr. P. Gritzmann | Dr. René Brandenberg

Multiple-Choice-Blatt

Exercise MC 1.1

	true	false
a) The euclidean algorithm for the determination of $\gcd(\alpha, \beta)$ is non-polynomial as the running time depends on α and β.	□	□
b) By studying the polynomiality of algorithms we may assume w.l.o.g. that any rational input-data is coprime.	□	□
c) Let $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$. Then the decision problem: "Does there exist an integer solution x to the linear inequality system $Ax \le b$?" is $\mathbb{NP}$-complete.	□	□

Exercise MC 1.2

	true	false
a) Let $Ax = b$ be a system of linear Diophantine equations with integer A, b such that the homogenous system $Ax = 0$ has non-trivial solutions. Then $Ax = b$ is feasible.	□	□
b) Let $Ax = b$ be a feasible system of linear Diophantine equations with integer A, b, s.t. $Ax = 0$ has non-trivial solutions. Then $Ax = b$ has infinitely many solutions.	□	□
c) Every system of linear Diophantine equations has no, one or infinitly many solutions.	□	□
d) If A, b are rational every solution of $Ax = b$ is rational.	□	□
e) If A, b are integer every solution of $Ax = b$ is integer.	□	□
f) If A, b are integer every rational solution of $Ax = b$ is integer.	□	□
g) The Hermite normal form of a one-row matrix $(a_1, \dots, a_m)$ is $(\gcd(a_1, \dots, a_m), 0, \dots, 0)$.	□	□
h) The Hermite normal form of the incidence matrix of a connected bipartite graph (after removing a redundand row) with n vertices and m edges has the form $(I_{n-1}, 0)$.	□	□

Exercise MC 1.3

Let $P = \{x \in \mathbb{R}^n : Ax \le b\}$, where $A \in \mathbb{R}^{n\times n}$ is square and $b \in \mathbb{R}^n$. Then P is unbounded.

true □ **false** □

Please turn over.

Exercise MC 1.4

In which of the following cases is the integer hull P_I of a polyhedron P again a polyhedron?

always □ **if** $\boldsymbol{P}$ **is bounded** □ **if** $\boldsymbol{P}$ **is rational** □ **if** $\boldsymbol{P}$ **is integer** □

if $\boldsymbol{P}$ **is a cone** □ **if** $\boldsymbol{P}$ **is line-free** □ **if** $\boldsymbol{P}$ **has finitely many vertices** □

Exercise MC 1.5

	true	false
a) There exist non-rational polyhedra P such that $P = P_I$.	□	□
b) If $P = \operatorname{conv}\{v_1, \ldots, v_k\}$ is a polytope with $v_1, \ldots, v_k \in \mathbb{Z}^n$ then $P = P_I$.	□	□
c) If $P = \operatorname{conv}\{v_1, \ldots, v_k\} + \operatorname{pos}\{z_1, \ldots, z_l\}$ is a polyhedron such that $v_1, \ldots, v_k$, $z_1, \ldots, z_l \in \mathbb{Z}^n$ then $P = P_I$.	□	□
d) If $P = \{x : Ax \leq b\}$ is a polyhedron with integral A, b then $P = P_I$.	□	□

Exercise MC 1.6

	true	false
a) If a lattice is generated by k vectors, then it is a linear transformation of $\mathbb{Z}^k$.	□	□
b) Any sublattice of $\mathbb{Z}^n$, which is generated by $k \leq n$ vectors, is a linear transformation of $\mathbb{Z}^k$.	□	□
c) Set inclusion defines a partial order on any family of sublattices of $\mathbb{Z}^k$, $k \in \mathbb{N}$.	□	□

Exercise MC 1.7

	true	false
a) If A is totally unimodular then $\begin{pmatrix} A & I \end{pmatrix}$ is totally unimodular.	□	□
b) If $\begin{pmatrix} A & I \end{pmatrix}$ is unimodular then A is totally unimodular. totally unimodular.	□	□
c) If $\begin{pmatrix} A & I \end{pmatrix}$ is unimodular then $\begin{pmatrix} A & I \end{pmatrix}$ totally unimodular.	□	□
d) The incidence matrix of a digraph is totally unimodular.	□	□
e) The Hermite normal form of a quadtratic unimodular matrix is again unimodular.	□	□
f) Any square submatrix of a totally unimodular matrix is regular.	□	□
g) The incidence matrix of a bipartite graph on n vertices always has rank at most $n-1$.	□	□
h) The incidence matrix of a graph is totally unimodular, iff the graph has no circuits.	□	□

. . . continued on next page . . .

i) Because of the total unimodularity of the incidence matrix of a bipartite graph, Koenig's theorem, that the maximal stable set is of the same cardinality than the minimal edge cover, follows directly from LP duality. ☐ ☐

Exercise MC 1.8

Let S_G be the incidence matrix of an arbitrary undirected graph with m edges. Then the vertices x^i, $i \in [l]$ of the corresponding perfect-matching polytope $\{x : S_G x = \mathbb{1}, x \geq 0\}$ may have arbitrary rational coordinates.

true ☐ **false** ☐

Exercise MC 1.9

Let $G = (V, E)$ a digraph with $V = \{v_1, \dots, v_n\}$ and $|E| = m$ as well as

$$P = \{x \in \mathbb{R}^m : \hat{S}_G x = u^{n-1},\ x \geq 0\},$$

where $\hat{S}_G$ denotes the incidence matrix of G, after deleting the first row.

Then the vertices of the polyhedron P correspond to v_1, v_n-pathes and they are all non-degenerate.

true ☐ **false** ☐

Exercise MC 1.10

Let $\text{pos}\{v_1, \dots, v_k\}$ with $v_i \in \mathbb{Z}^n$, $i \in [k]$ be a cone. For a minimal Hilbert-basis H of the cone we always have

	true	**false**
a) H is unique.	☐	☐
b) $\lvert H\rvert \leq 2n + 1$	☐	☐
c) $H \subset \text{conv}\{v_1, \dots, v_k\}$.	☐	☐
d) $H \subset \sum_{i=1}^{k} [0, 1] v_i = \left\{ \sum_{i=1}^{k} \lambda_i v_i : 0 \leq \lambda_i \leq 1 \right\}$.	☐	☐
e) If $v_i \in \mathbb{Q}^n$, $i \in [k]$, then there is no Hilbert basis for P of finite size.	☐	☐

Exercise MC 1.11

Let $P = \{x : Ax \leq b\}$ a rational polyhedron. Which of the following statements are true?

	true	**false**
a) Theoretically, there exists an infinite number of R-cuts for P, but only a finite number of them is non-redundant.	☐	☐
b) If one could add all (infinetly many) R-cuts for P to the system $Ax \leq b$ at the same time then one would get an $\mathcal{H}$-description of P_I directly.	☐	☐
c) $\{x : Ax \leq \lfloor b \rfloor\}$ is the Chvatal-Gomory-closure of P.	☐	☐
d) There exists $k \in \mathbb{N}$, such that $P_k = P_I$.	☐	☐

Please turn over.

e) To compute the Chvatal-Gomory-closure of P, if P is pointed, it suffices to add all R-cuts, corresponding to elements of the Hilbert-bases of all normal cones at fractional vertices of P, to the system $Ax \leq b$. □ □

f) Every Gomory-cut for P is an R-cut corresponding to an element of the Hilbert-basis of the normal cone of a fractional vertex of P. □ □

g) If P is a rational polyhedron of Chvatal-Gomory rank k, then any proper R-cut at P_{k-1} belongs to an irredundant $\mathcal{H}$-presentation of P_I. □ □

h) In each step, the Gomory cutting plane algorithm always introduces a cutting hyperplane which meets an integral point of the polyhedron. □ □

i) If P is a rational polyhedron of Chvatal-Gomory rank k, then the Gomory cutting plane algorithm requieres at least k iterations. □ □

Exercise MC 1.12

Let P be a polytope and let P_1 be its Chvátal-Gomory closure. Which of the following statements are true for all P and corresponding P_1, which are false for at least one pair of P and corresponding P_1?

- □ $\dim(P_1) = \dim(P)$.
- □ If $P_1 \neq P$, then P_1 has strictly fewer vertices than P.
- □ If $P_1 \neq P$, then P_1 has strictly more facets than P.
- □ If $P_1 \neq P$, then the Chvátal-Gomory rank of P_1 is strictly less than the Chvátal-Gomory rank of P.

Exercise MC 1.13

Let $c, d, A, B, bintegral$ and (x^*, y^*) be an optimal solution to the linear relaxation for the mixed-integer linear program (MILP)

$$\max c^T x + d^T y, Ax + By \leq b, x \in \mathbb{Z}^n$$

and x^* fractional. Then

$$c^T x + d^T y \leq \lfloor c^T x^* \rfloor + d^T y^*$$

is a valid cut (i. e. it is fulfilled by all feasible solutions of the MILP). **true** □ **false** □