2006, Tag 1

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a real function. Prove or disprove each of the following statements.

(a) If f is continuous and range$(f) = \mathbb{R}$ then f is monotonic.

(b) If f is monotonic and range$(f) = \mathbb{R}$ then f is continuous.

(c) If f is monotonic and f is continuous then range$(f) = \mathbb{R}$.

(20 points)

Solution. (a) False. Consider function $f(x) = x^3 - x$. It is continuous, range$(f) = \mathbb{R}$ but, for example, $f(0) = 0$, $f(\frac{1}{2}) = -\frac{3}{8}$ and $f(1) = 0$, therefore $f(0) > f(\frac{1}{2})$, $f(\frac{1}{2}) < f(1)$ and f is not monotonic.

(b) True. Assume first that f is non-decreasing. For an arbitrary number a, the limits $\lim\limits_{a-} f$ and $\lim\limits_{a+} f$ exist and $\lim\limits_{a-} f \le \lim\limits_{a+} f$. If the two limits are equal, the function is continuous at a. Otherwise, if $\lim\limits_{a-} f = b < \lim\limits_{a+} f = c$, we have $f(x) \le b$ for all $x < a$ and $f(x) \ge c$ for all $x > a$; therefore range$(f) \subset (-\infty, b) \cup (c, \infty) \cup \{f(a)\}$ cannot be the complete $\mathbb{R}$.

For non-increasing f the same can be applied writing reverse relations or $g(x) = -f(x)$.

(c) False. The function $g(x) = \arctan x$ is monotonic and continuous, but range$(g) = (-\pi/2, \pi/2) \ne \mathbb{R}$.

2007, Tag 2

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that for any $c > 0$, the graph of f can be moved to the graph of cf using only a translation or a rotation. Does this imply that $f(x) = ax + b$ for some real numbers a and b?

Solution. No. The function $f(x) = e^x$ also has this property since $ce^x = e^{x+\log c}$.

Problem 3. Let C be a nonempty closed bounded subset of the real line and $f : C \to C$ be a nondecreasing continuous function. Show that there exists a point $p \in C$ such that $f(p) = p$.

(A set is closed if its complement is a union of open intervals. A function g is nondecreasing if $g(x) \le g(y)$ for all $x \le y$.)

Solution. Suppose $f(x) \ne x$ for all $x \in C$. Let $[a, b]$ be the smallest closed interval that contains C. Since C is closed, $a, b \in C$. By our hypothesis $f(a) > a$ and $f(b) < b$. Let $p = \sup\{x \in C : \quad f(x) > x\}$. Since C is closed and f is continuous, $f(p) \ge p$, so $f(p) > p$. For all $x > p$, $x \in C$ we have $f(x) < x$. Therefore $f(f(p)) < f(p)$ contrary to the fact that f is non-decreasing.

2006,2

Problem 2. Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that for any real numbers $a < b$, the image $f([a, b])$ is a closed interval of length $b - a$.

Solution. The functions $f(x) = x + c$ and $f(x) = -x + c$ with some constant c obviously satisfy the condition of the problem. We will prove now that these are the only functions with the desired property.

Let f be such a function. Then f clearly satisfies $|f(x) - f(y)| \le |x - y|$ for all x, y; therefore, f is continuous. Given x, y with $x < y$, let $a, b \in [x, y]$ be such that $f(a)$ is the maximum and $f(b)$ is the minimum of f on $[x, y]$. Then $f([x, y]) = [f(b), f(a)]$; hence

$$y - x = f(a) - f(b) \le |a - b| \le y - x$$

This implies $\{a, b\} = \{x, y\}$, and therefore f is a monotone function. Suppose f is increasing. Then $f(x) - f(y) = x - y$ implies $f(x) - x = f(y) - y$, which says that $f(x) = x + c$ for some constant c. Similarly, the case of a decreasing function f leads to $f(x) = -x + c$ for some constant c.

1998, 1

Problem 3. Let $f(x) = 2x(1-x)$, $x \in \mathbb{R}$. Define

$$f_n = \overbrace{f \circ \ldots \circ f}^{n}.$$

a) *(10 points)* Find $\lim_{n\to\infty} \int_0^1 f_n(x)dx$.

b) *(10 points)* Compute $\int_0^1 f_n(x)dx$ for $n = 1, 2, \ldots$.

Solution. a) Fix $x = x_0 \in (0,1)$. If we denote $x_n = f_n(x_0)$, $n = 1, 2, \ldots$ it is easy to see that $x_1 \in (0, 1/2]$, $x_1 \le f(x_1) \le 1/2$ and $x_n \le f(x_n) \le 1/2$ (by induction). Then $(x_n)_n$ is a bounded non-decreasing sequence and, since $x_{n+1} = 2x_n(1-x_n)$, the limit $l = \lim_{n\to\infty} x_n$ satisfies $l = 2l(1-l)$, which implies $l = 1/2$. Now the monotone convergence theorem implies that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 1/2.$$

b) We prove by induction that

$$f_n(x) = \frac{1}{2} - 2^{2^n-1}\left(x - \frac{1}{2}\right)^{2^n} \tag{1}$$

holds for $n = 1, 2, \ldots$. For $n = 1$ this is true, since $f(x) = 2x(1-x) = \frac{1}{2} - 2(x - \frac{1}{2})^2$. If (1) holds for some $n = k$, then we have

$$\begin{aligned} f_{k+1}(x) &= f_k\left(f(x)\right) = \tfrac{1}{2} - 2^{2^k-1}\left(\left(\tfrac{1}{2} - 2(x-\tfrac{1}{2})^2\right) - \tfrac{1}{2}\right)^{2^k} \\ &= \tfrac{1}{2} - 2^{2^k-1}\left(-2(x-\tfrac{1}{2})^2\right)^{2^k} \\ &= \tfrac{1}{2} - 2^{2^{k+1}-1}(x-\tfrac{1}{2})^{2^{k+1}} \end{aligned}$$

which is (2) for $n = k+1$.

Using (1) we can compute the integral,

$$\int_0^1 f_n(x)dx = \left[\frac{1}{2}x - \frac{2^{2^n-1}}{2^n+1}\left(x - \frac{1}{2}\right)^{2^n+1}\right]_{x=0}^1 = \frac{1}{2} - \frac{1}{2(2^n+1)}.$$

Problem 6. Let $f : [0,1] \to \mathbb{R}$ be a continuous function with the property that for any x and y in the interval,

$$xf(y) + yf(x) \le 1.$$

a) *(15 points)* Show that

$$\int_0^1 f(x)dx \le \frac{\pi}{4}.$$

b) *(5 points)* Find a function, satisfying the condition, for which there is equality.

Solution Observe that the integral is equal to

$$\int_0^{\frac{\pi}{2}} f(\sin\theta)\cos\theta d\theta$$

and to

$$\int_0^{\frac{\pi}{2}} f(\cos\theta)\sin\theta d\theta$$

So, twice the integral is at most

$$\int_0^{\frac{\pi}{2}} 1 d\theta = \frac{\pi}{2}.$$

Now let $f(x) = \sqrt{1-x^2}$. If $x = \sin\theta$ and $y = \sin\phi$ then

$$xf(y) + yf(x) = \sin\theta\cos\phi + \sin\phi\cos\theta = \sin(\theta+\phi) \le 1.$$

2005, 1

Problem 3. Let $f:\mathbb{R}\to[0,\infty)$ be a continuously differentiable function. Prove that

$$\left|\int_0^1 f^3(x)\,dx - f^2(0)\int_0^1 f(x)\,dx\right| \le \max_{0\le x\le 1}|f'(x)|\left(\int_0^1 f(x)\,dx\right)^2.$$

Solution 1. Let $M=\max_{0\le x\le 1}|f'(x)|$. By the inequality $-M\le f'(x)\le M$, $x\in[0,1]$ it follows:

$$-Mf(x)\le f(x)f'(x)\le Mf(x),\ x\in[0,1].$$

By integration

$$-M\int_0^x f(t)\,dt \le \frac{1}{2}f^2(x)-\frac{1}{2}f^2(0)\le M\int_0^x f(t)\,dt,\ x\in[0,1]$$

$$-Mf(x)\int_0^x f(t)\,dt \le \frac{1}{2}f^3(x)-\frac{1}{2}f^2(0)f(x)\le Mf(x)\int_0^x f(t)\,dt,\ x\in[0,1].$$

Integrating the last inequality on $[0,1]$ it follows that

$$-M\left(\int_0^1 f(x)dx\right)^2 \le \int_0^1 f^3(x)\,dx - f^2(0)\int_0^1 f(x)\,dx \le M\left(\int_0^1 f(x)dx\right)^2 \Leftrightarrow$$

$$\left|\int_0^1 f^3(x)\,dx - f^2(0)\int_0^1 f(x)\,dx\right| \le M\left(\int_0^1 f(x)\,dx\right)^2.$$

Solution 2. Let $M=\max_{0\le x\le 1}|f'(x)|$ and $F(x)=-\int_x^1 f$; then $F'=f$, $F(0)=-\int_0^1 f$ and $F(1)=0$. Integrating by parts,

$$\int_0^1 f^3=\int_0^1 f^2\cdot F'=[f^2F]_0^1-\int_0^1 (f^2)'F=$$

$$=f^2(1)F(1)-f^2(0)F(0)-\int_0^1 2Fff'=f^2(0)\int_0^1 f-\int_0^1 2Fff'.$$

Then

$$\left|\int_0^1 f^3(x)\,dx - f^2(0)\int_0^1 f(x)\,dx\right|=\left|\int_0^1 2Fff'\right|\le\int_0^1 2Ff|f'|\le M\int_0^1 2Ff=M\cdot[F^2]_0^1=M\left(\int_0^1 f\right)^2.$$

1998, 1

Problem 4. *(20 points)* The function $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and satisfies $f(0) = 2$, $f'(0) = -2$ and $f(1) = 1$. Prove that there exists a real number $\xi \in (0, 1)$ for which

$$f(\xi) \cdot f'(\xi) + f''(\xi) = 0.$$

Solution. Define the function

$$g(x) = \frac{1}{2} f^2(x) + f'(x).$$

Because $g(0) = 0$ and

$$f(x) \cdot f'(x) + f''(x) = g'(x),$$

it is enough to prove that there exists a real number $0 < \eta \le 1$ for which $g(\eta) = 0$.

a) If f is never zero, let

$$h(x) = \frac{x}{2} - \frac{1}{f(x)}.$$

Because $h(0) = h(1) = -\frac{1}{2}$, there exists a real number $0 < \eta < 1$ for which $h'(\eta) = 0$. But $g = f^2 \cdot h'$, and we are done.

b) If f has at least one zero, let z_1 be the first one and z_2 be the last one. (The set of the zeros is closed.) By the conditions, $0 < z_1 \le z_2 < 1$.

The function f is positive on the intervals $[0, z_1)$ and $(z_2, 1]$; this implies that $f'(z_1) \le 0$ and $f'(z_2) \ge 0$.

Then $g(z_1) = f'(z_1) \le 0$ and $g(z_2) = f'(z_2) \ge 0$, and there exists a real number $\eta \in [z_1, z_2]$ for which $g(\eta) = 0$.

Remark. For the function $f(x) = \frac{2}{x+1}$ the conditions hold and $f \cdot f' + f''$ is constantly 0.

2005, 2

Problem 4. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is three times differentiable, then there exists a real number $\xi \in (-1, 1)$ such that

$$\frac{f'''(\xi)}{6} = \frac{f(1) - f(-1)}{2} - f'(0).$$

Solution 1. Let

$$g(x) = -\frac{f(-1)}{2} x^2(x-1) - f(0)(x^2 - 1) + \frac{f(1)}{2} x^2(x+1) - f'(0)x(x-1)(x+1).$$

It is easy to check that $g(\pm 1) = f(\pm 1)$, $g(0) = f(0)$ and $g'(0) = f'(0)$.

Apply Rolle's theorem for the function $h(x) = f(x) - g(x)$ and its derivatives. Since $h(-1) = h(0) = h(1) = 0$, there exist $\eta \in (-1, 0)$ and $\vartheta \in (0, 1)$ such that $h'(\eta) = h'(\vartheta) = 0$. We also have $h'(0) = 0$, so there exist $\varrho \in (\eta, 0)$ and $\sigma \in (0, \vartheta)$ such that $h''(\varrho) = h''(\sigma) = 0$. Finally, there exists a $\xi \in (\varrho, \sigma) \subset (-1, 1)$ where $h'''(\xi) = 0$. Then

$$f'''(\xi) = g'''(\xi) = -\frac{f(-1)}{2} \cdot 6 - f(0) \cdot 0 + \frac{f(1)}{2} \cdot 6 - f'(0) \cdot 6 = \frac{f(1) - f(-1)}{2} - f'(0).$$

Solution 2. The expression $\dfrac{f(1) - f(-1)}{2} - f'(0)$ is the divided difference $f[-1, 0, 0, 1]$ and there exists a number $\xi \in (-1, 1)$ such that $f[-1, 0, 0, 1] = \dfrac{f'''(\xi)}{3!}$.