Discrete Optimization (MA 3502)

Prof. Dr. P. Gritzmann | Dipl.-Math. Viviana Ghiglione | Dr. M. Ritter

Exercise Sheet 1

Exercise 1.1 (Rounding Solutions)
Let $a \in \mathbb{Z}^2$, $\beta \in \mathbb{Z}$ and consider the polyhedron

$$P(a,\beta) := \left\{x \in \mathbb{R}^2 : a^T x \leq \beta, x \geq 0\right\}$$

and its *integer hull* $I(P)$ defined as

$$I(P) := \mathrm{conv}(P \cap \mathbb{Z}^2).$$

We want to solve the integer linear program $\max_{x \in P(a,\beta)} c^T x$ for some integral objective vector $c \in \mathbb{Z}^2$. Let x^* denote an optimal integer solution to this ILP and let $x' \in \mathbb{R}^2$ be an optimal solution of the LP relaxation, i. e. the ILP without the integrality constraints. In this exercise, we investigate the strategy of simply rounding x' down componentwise (denoted by $\lfloor x' \rfloor$) to get an integer solution.

a) Is $x' = x^*$ is possible? Either give an example or disprove the statement!

b) Is it possible that $x' \neq x^*$, but $\lfloor x' \rfloor = x^*$? Either give an example or disprove the statement!

c) Show that $\lfloor x' \rfloor \in I(P(a,\beta))$ holds if $a, \beta \geq 0$.

d) Give an example that shows that x^* and $\lfloor x' \rfloor$ need not be "close" (both with respect to Euclidean distance and with respect to the objective value).

Exercise 1.2 (Simplex Tableau Revisited)
Consider the following linear program:

$$\begin{aligned} \max\ 5x_1 + 6x_2 & \\ 3x_1 + 5x_2 &\leq 15 \\ 3x_1 - 5x_2 &\leq 0 \end{aligned}$$

a) Sketch the feasible region for the above LP and guess an optimal solution.

b) Write down the dual of the above LP.

c) Prove optimality of your primal solution by devising a corresponding dual optimal solution.

d) Show how to compute the solution using the dual simplex method in tableau form.

e) The linear program is now modified by appending the inequality $2x_1 + x_2 \leq 6$. Show that this modification invalidates your current primal solution. Use the dual simplex method in tableau form to compute a new dual and primal solution. Can you re-use some of the work you did in the previous part of this exercise?

Please turn over.

Exercise 1.3 (The Matching Polytope)

Let $G = (V, E)$ be a graph on n vertices and m edges. Let S_G denote the node-edge incidence matrix of G and let $\mathcal{M}(G)$ denote the *matching polytope* of G, i. e. the convex hull of all feasible matchings of G:

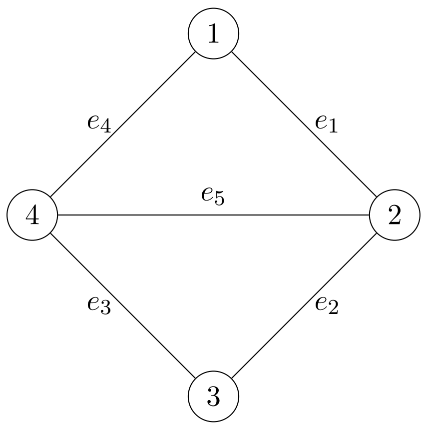
$$\mathcal{M}(G) := \operatorname{conv}(\{x \in \{0,1\}^m : S_G x \leq \mathbb{1}\})$$

Further, let $P = \{x \in \mathbb{R}^m : S_G x \leq \mathbb{1}, x \geq 0\}$ denote the LP relaxation of $\mathcal{M}(G)$.

a) Determine the dimension $\dim(\mathcal{M}(G))$.

b) Show that all inequalities of the form $x_e \geq 0$ define a facet of $\mathcal{M}(G)$.

In the following, consider the example of the graph depicted below.



c) Determine the outer normal cone of P at the vertex $x^* = (1/2, 0, 0, 1/2, 1/2)^T$.

d) Consider the vertex $x' := (0, 1/2, 1/2, 0, 1/2)$. Show that the inequality

$$x_2 + x_3 + x_5 \leq 1$$

cuts off the fractional point x', but is a valid inequality for $\mathcal{M}(G)$.