Discrete Optimization (MA 3502)

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Exercise Sheet 1

Exercise 1.1 (Rounding Solutions)
Let $a \in \mathbb{Z}^2$, $\beta \in \mathbb{Z}$ and consider the polyhedron

$$P(a, \beta) := \left\{x \in \mathbb{R}^2 : a^T x \leq \beta, x \geq 0\right\}$$

and its *integer hull* $(P)_I$ defined as

$$(P)_I := \operatorname{conv}(P \cap \mathbb{Z}^2).$$

We want to solve the integer linear program $\max_{x \in P(a,\beta)} c^T x$ for some integral objective vector $c \in \mathbb{Z}^2$. Let x^* denote an optimal integer solution to this ILP and let $x' \in \mathbb{R}^2$ be an optimal solution of the LP relaxation, i. e. the ILP without the integrality constraints. In this exercise, we investigate the strategy of simply rounding x' down componentwise (denoted by $\lfloor x' \rfloor$) to get an integer solution.

a) Is $x' = x^*$ possible? Either give an example or disprove the statement!

b) Is it possible that $x' \neq x^*$, but $\lfloor x' \rfloor = x^*$? Either give an example or disprove the statement!

c) Show that $\lfloor x' \rfloor \in (P(a, \beta))_I$ holds if $a, \beta \geq 0$.

d) Give an example that shows that x^* and $\lfloor x' \rfloor$ need not be "close" (both with respect to Euclidean distance and with respect to the objective value).

Answer to Exercise 1.1

a) Take $c = (1, 0)^T$, $a = (1, 1)^T$ and $\beta = 1$. Then, $x' = (1, 0)^T = x^*$, thus the relaxation readily yields an integral solution.

b) Take $c = (1, 0)^T$, $a = (2, 3)^T$ and $\beta = 3$. Then the relaxed solution is $x' = (1.5, 0)^T$ and rounding yields $\lfloor x' \rfloor = (1, 0)^T$, which is also the integral solution.

c) Let x' be any feasible solution to the LP relaxation. Then

$$a^T \lfloor x' \rfloor \leq a^T x' \leq \beta,$$

thus $\lfloor x' \rfloor$ is still a feasible solution to the LP. As it is also integral, it is contained in $(P(a, \beta))_I$.

d) Take $c = (21, 11)^T$, $a = (7, 4)^T$ and $\beta = 13$. Then, $x' = (\frac{13}{7}, 0)^T$, $\lfloor x' \rfloor = (1, 0)^T$, but $x^* = (0, 3)^T$. With $c^T x^* = 33$ and $c^T \lfloor x' \rfloor = 21$, those solutions are not very close to each other.

Exercise 1.2 (The Assignment Problem)
Let $T = \{t_1, \ldots, t_k\}$ be a finite set of *tasks* that need to be assigned to a set of *workers* $W = \{w_1, \ldots, w_p\}$. Each worker is able to each task, but depending on how well the worker w_i is trained to do a specific task t_j a *cost* c_{ij} is incurred.

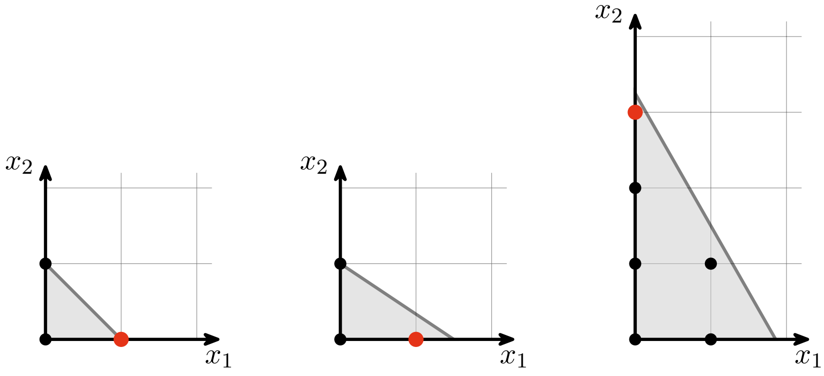

Figure 1: Illustration of the examples.

a) Assuming $p \geq k$, design an integer linear program that models the assignment of tasks to workers with the objective of minimizing the total cost of the assignment. Each task needs to be assigned to a worker, a task cannot be subdivided and a worker is only able to do one task.

b) In addition to the cost there is now a time requirements T_j for each task t_j and a time budget B_i for each worker w_i. Design an integer linear program that models the assignment of tasks to workers with the objective of minimizing the total cost. Each task needs to be assigned, a task cannot be subdivided. Each worker may receive more than one task, but their total time requirement must not exceed the worker's budget.

c) Assuming $p \geq k$, design an integer linear program that models the assignment of tasks to workers with the objective of minimizing the maximum cost that is incurred among all worker-task-pairs. Each task needs to be assigned to a worker, a task cannot be subdivided and a worker is only able to do one task.

Answer to Exercise 1.2

a) We use binary variables x_{ij} modelling the assignment of tasks to workers, specifically

$$x_{ij} = \begin{cases} 1, & \text{if task } t_j \text{ is assigned to worker } w_i \\ 0, & \text{otherwise.} \end{cases}$$

Then the problem can be expressed through the following integer linear program:

$$\min \sum_{i=1}^{p} \sum_{j=1}^{k} c_{ij} x_{ij} \tag{1}$$

$$\sum_{i=1}^{p} x_{ij} = 1 \quad \text{for each task } t_j \in T \tag{2}$$

$$\sum_{j=1}^{k} x_{ij} \leq 1 \quad \text{for each worker } w_i \in W \tag{3}$$

$$x \in \{0,1\}^{p \times k} \tag{4}$$

While Equation (1) models the cost of an assignment, the constraints Equation (2) and Equation (3) reflect the conditions that each task is fully assigned and that each worker can handle at most one task, respectively. Finally, Equation (4) requires the variables to be binary.

b) In contrast to the first model, we now remove the constraint Equation (3) that each worker may only handle one task and replace it with a constraint on the time budget Equation (5). The new model then looks like this:

$$\begin{aligned}
\min \; & \sum_{i=1}^{p}\sum_{j=1}^{k} c_{ij}x_{ij} \\
& \sum_{i=1}^{p} x_{ij} = 1 \quad \text{for each task } t_j \in T \\
& \sum_{j=1}^{k} T_j x_{ij} \leq B_i \quad \text{for each worker } w_i \in W \qquad (5) \\
& x \in \{0,1\}^{p\times k}
\end{aligned}$$

c) The problem is now basically the same as in the first part, but the objective needs to be changed. Let us write down the new model first and then have a closer look at the new objective:

$$\begin{aligned}
\min \; & \max \{c_{ij}x_{ij} : i \in [p], j \in [k]\} \qquad (6) \\
& \sum_{i=1}^{p} x_{ij} = 1 \quad \text{for each task } t_j \in T \\
& \sum_{j=1}^{k} x_{ij} \leq 1 \quad \text{for each worker } w_i \in W \\
& x \in \{0,1\}^{p\times k}
\end{aligned}$$

Unfortunately, the new objective is not a linear expression anymore. To overcome this, we will introduce a new (continuous) variable z whose value will be the maximum that we are trying to minimize. We can achieve this by adding some constraints to our model:

$$\begin{aligned}
\min \; & z \qquad (7) \\
& \sum_{i=1}^{p} x_{ij} = 1 \quad \text{for each task } t_j \in T \\
& \sum_{j=1}^{k} x_{ij} \leq 1 \quad \text{for each worker } w_i \in W \\
& z \geq c_{ij}x_{ij} \quad \text{for all } i \in [p],\ j \in [k] \qquad (8) \\
& x \in \{0,1\}^{p\times k} \\
& z \geq 0
\end{aligned}$$

The constraints Equation (8) ensure that z assumes at least the value of the maximum among all assignment costs, while the objective Equation (7) ensures that z will not get larger than that, hence the model is doing exactly what it is supposed to do and we have replaced Equation (6) by some linear expressions.

Exercise 1.3 (The Traveling Salesman Problem)
The traveling salesman problem on the complete undirected graph $G = (V, E)$ on the vertex set $V = [n]$ with edge weights $d : E \to \mathbb{N}_0$ asks for a tour of the nodes ("cities") that visits every node

exactly once and has minimum total weight ("tour length"). Consider the following integer linear program:

$$\begin{aligned}
\min \sum_{e\in E} d_e x_e & \\
\sum_{e\in\delta(v)} x_e = 2 & \quad \text{for all } v\in V \\
\sum_{e\in E(S)} x_e \le |S|-1 & \quad \text{for all } \emptyset \neq S \subsetneq V \qquad (9)\\
x\in\{0,1\}^{|E|} &
\end{aligned}$$

a) Give an interpretation of the binary variables and of the constraints in the above program.

b) Consider the program without the *subtour elimination constraints* Equation (9) and show that each feasible solution of that program corresponds to a collection of node disjoint cycles in G.

c) Prove that the integer linear program (with all constraints) correctly models the traveling salesman problem, i. e. there is a bijection between the set of traveling salesman tours on G and the set of feasible integer solutions of the ILP.

d) Show that the subtour elimination constraints may be replaced by the following *cut-set constraints*:

$$\sum_{e\in\delta(S)} x_e \ge 1 \quad \text{for all } \emptyset \neq S \subsetneq V$$

Answer to Exercise 1.3

a) The binary variable x_e indicates whether an edge $e \in E$ is used as part of a solution ($x_e = 1$) or not ($x_e = 0$). The objective function is then simply the sum of lengths over all edges that are part of the solution, hence it correctly represents the length of the tour. The first constraint models the condition that for each node exactly two incident edges need to be part of the tour, i. e. each node has to be visited exactly once. The second class of constraints, known as *subtour elimination constraints*, forbid any cycles in the solution unless they contain the whole node set: Consider a cycle that is part of the solution x, but whose node set S is not equal to V. This cycle then contains $|S|$ edges, thus the subtour elimination constraint corresponding to that particular set S would be violated. Finally, x is restricted to binary coordinate values.

b) Let x^* be a feasible solution of the program without the subtour elimination constraints. Due to the first class of inequalities, the degree of each node with respect to the edge set $E^* := \{e \in E : x^*_e = 1\}$ defined by x^* is precisely 2. Consider some node $v_1 \in V$, then by following one of the edges from E^* that is incident to v_1 we reach a different node v_2. We again follow the edge in E^* that is incident to v_2 and that has not yet been used and construct a path. As there are only finitely many nodes, this process will end at some point, meaning we reach a node that does not have any unused edge any more. As the degree of each node is precisely 2, the only node with that property is v_1, thus v_1 is contained in a cycle. As this cycle uses all the edges incident to the nodes within the cycle, it cannot share a node with any other cycle, thereby proving the claim.

c) For a traveling salesman tour, setting $x_e = 1$ for all edges e in the tour and 0 otherwise gives us a feasible solution. The degree contraints are obviously satisfied, as a tour is also a cycle, and the subtour elimination constraints are satisfied as otherwise we would have a subtour in the solution. Conversely, for a feasible integer solution x^* of the linear program, the edge set

$E^* := \{e \in E : x^*_e = 1\}$ defines a tour: By the previous part it comprises of node disjoint cycles. Assume more than one cycle is contained in the solution, consider the node set S of one of these cycles. The corresponding subtour elimination constraint would then be violated, thus the solution is one cycle that contains all nodes in V and consequently a traveling salesman tour.

d) Let x^* be a feasible solution of our original integer linear program. We claim that x^* also satisfies all cut set constraints. Assume there is some vertex x^* of the feasible region and some set $S \subsetneq V$ such that x^* does not satisfy the cut-set constraint corresponding to S (we may assume x^* is a vertex, because if every vertex satisfies the cut-set constraint then by convexity every feasible point will). In particular, $x^* \in \{0,1\}^{|E|}$, thus $x^*_e = 0$ for all $e \in \delta(S)$. As we have already seen that x^* corresponds to a feasible traveling salesman tour, the solution consists of a single connected component, thus S is either $\emptyset$ or V, both contradicting our assumption.

   To see the reverse, consider a vertex x^* of the feasible region defined by replacing all constraints of the type Equation (9)) by their cut-set counterpart. Assume there was some set S such that x^* violated the subtour elimination constraint for S. As x^* is binary (remember we assumed x^* to be a vertex!), that means the set S contains at least $|S|$ edges $e \in E(S)$ with $x^*_e = 1$. By the degree constraints this edge set comprises of node disjoint cycle (see above argument). If we now choose S' as the node set of one of these cycles, then the cut-set constraint for S' would be violated, contradicting our original assumption.

Exercise 1.4
Let P and Q be polytopes in $\mathbb{R}^n$ where $n \geq 1$. Prove or disprove: $(P \setminus Q)_I = (P)_I \setminus (Q)_I$.

Answer to Exercise 1.4
The statement is false. The integer hull of a polyhedron needs to be convex, while the difference of two convex sets does not need to be convex. In $\mathbb{R}^1$, take $P = \{x : -1 \leq x \leq 1\}$, $Q = \{0\}$. Then

$$I(P \setminus Q) = \operatorname{conv}\{-1, 1\} = [-1, 1] \neq I(P) \setminus I(Q) = \operatorname{conv}\{-1, 0, 1\} \setminus \{0\} = [-1, 1] \setminus \{0\}.$$

Exercise 1.5
Let $P \subset \mathbb{R}^2$ be defined through the following $\mathcal{H}$-presentation:

$$-\sqrt{2}x_1 + x_2 \leq 0$$
$$x_1 - \sqrt{2}x_2 \leq 0$$

a) Sketch both P and the integral points contained in P. Take a guess at the integer hull $(P)_I := \operatorname{conv}(P \cap \mathbb{Z}^2)$ based on your sketch!

b) Show that $P = \operatorname{pos}\left\{(1, \sqrt{2})^T, (\sqrt{2}, 1)^T\right\}$ ($\mathcal{V}$-presentation).

c) Prove that $(P)_I = \{0\} \cup \operatorname{int}(P)$. You may use the following result without proof: For every line in $\mathbb{R}^2$ with irrational slope there exist integer points arbitrarily close to (and on both sides of) the line.

d) Is $(P)_I$ a polyhedron?

e) Let $Q := \left\{(-\frac{1}{2}, -\frac{1}{2})^T\right\} + P$. Show that the integer hull $(Q)_I$ of Q has an infinite number of vertices.

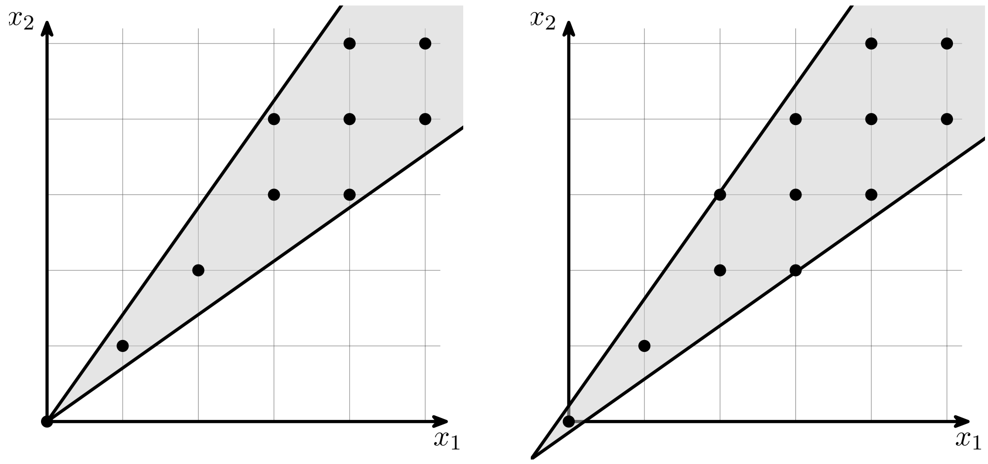

Figure 2: Illustration of P and Q and the integral points contained therein.

Answer to Exercise 1.5

a) The sketch in Figure 2 shows both P and the integral points contained in P. For the integer hull, see the rest of this problem.

b) The two inequalities defining P clearly intersect in 0, and as they do not define parallel lines, 0 is the only vertex of P. The outer normals of P at 0 are $(-\sqrt{2}, 1)^T$ and $(1, -\sqrt{2})^T$, thus the extreme rays are perpendicular to these and pointing towards the positive orthand. This yields the desired $\mathcal{V}$-presentation

$$P = \operatorname{pos}\left\{ \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \right\}.$$

c) **"$\subset$":** We start by showing $(P)_I \subset \{0\} \cup \operatorname{int}(P)$. Cleary, $(P)_I \subset P$, thus it suffices to show that 0 is the only integer point lying on the boundary of P. Assume otherwise, then the extreme ray that contained such a point would have rational slope (since it also contains 0). This clearly contradicts the irrationality of $\sqrt{2}$, thus proving that 0 is in fact the only integral point on the boundary of P.

"$\supset$": We show the converse, i. e. $\{0\} \cup \operatorname{int}(P) \subset (P)_I$, in two steps. Clearly, $0 \in (P)_I$, hence we only need to consider some $x \in \operatorname{int}(P)$, $x \neq 0$.

i) If $\operatorname{pos}\{x\}$ contains a rational point, then it also contains some integer point p (multiply with the denominator), and by multiplying with some suitably large $n \in \mathbb{N}$ we can assume that $x \in \operatorname{conv}\{0, np\}$ (meaning x lies on the line segment connecting 0 and np). Since $\{0, np\} \subset P \cap \mathbb{Z}^2$ we have $\operatorname{conv}\{0, np\} \subset (P)_I$ and thus $x \in (P)_I$.

ii) If $\operatorname{pos}\{x\}$ contains no rational point, we denote the extreme rays of P by $r_1 := \operatorname{pos}\left\{(1, \sqrt{2})^T\right\}$ and $r_2 := \left\{(\sqrt{2}, 1)^T\right\}$, respectively, and consider the rays $x + r_1$ and $x + r_2$. Note that $x + r_1 \subset \operatorname{int}(P)$, $x + r_2 \subset \operatorname{int}(P)$, and $\operatorname{pos}\{x\} \in \operatorname{pos}\{x + r_1, x + r_2\}$.

Using the statement in the hint, we know that there are integer points arbitrarily close to $x + r_1$ and $x + r_2$, respectively. Thus we can choose integer points p_1, p_2 that are close enough to $x + r_1$ and $x + r_2$ to still be contained in $\operatorname{int}(P)$ and such that they are on opposing sides of $\operatorname{pos}\{x\}$. Then, $\operatorname{pos}\{x\} \subset \operatorname{pos}\{p_1, p_2\} \subset (P)_I$, showing $x \in (P)_I$.

Remark: Suppose all vertices of a polyhedron P to be integer, and consider an LP over P with a finite optimum. We know that the LP has a vertex solution, which means it *has an integer solution, in this special case. Still, P needs not to be equal to $(P)_I$, as this exercise shows.*

d) The integer hull $(P)_I$ is not a polyhedron. Assume otherwise, then $(P)_I$ could be written as $(P)_I = \operatorname{conv} V + \operatorname{pos} R$ with a finite set of vertices V and of extreme rays R. However, this representation clearly is closed, in contrast to $(P)_I = \{0\} \cup \operatorname{int}(P)$ as shown in the previous problem, a contradiction.

e) First note that $(P)_I \neq (Q)_I$, so the result of the preceding problem cannot simply be transferred. The reason for this is that the "surrounding cone" that defines Q (and thus $(Q)_I$) has been shifted such that its apex is not an integral point anymore.

It is easy to see that $p_0 := 0$ is a vertex of $(Q)_I$, see Figure 2 for an illustration. Suppose that $(Q)_I$ has only finitely many vertices $v_1, \ldots, v_k$. Since $(Q)_I$ is unbounded there must be an $n \in \mathbb{N}$ such that v_n has only one neighboring vertex (the "last vertex at the verge of inifinity"). Let $r := \operatorname{pos}\left\{(1, \sqrt{2})^T\right\}$ and consider the rays $s_1 := -(\frac{1}{2}, \frac{1}{2}) + r$ and $s_2 := v_n + r$. Notice that v_n is not lying on s_1, because otherwise that line would contain two rational points which would mean it had a rational slope. We again use the hint given for one of previous problems: There must be an integer point between s_1 and s_2, and if that was contained in $(Q)_I$ then v_n could not have been a vertex, a contradiction. Thus $(Q)_I$ cannot be described by any finite number of vertices and extreme rays.