Technische Universität München, Zentrum Mathematik
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Discrete Optimization (MA 3502)

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Exercise Sheet 1

Exercise 1.1 (Rounding Solutions)
Let $a \in \mathbb{Z}^2$, $\beta \in \mathbb{Z}$ and consider the polyhedron

$$P(a, \beta) := \left\{x \in \mathbb{R}^2 : a^T x \leq \beta, x \geq 0\right\}$$

and its *integer hull* $(P)_I$ defined as

$$(P)_I := \operatorname{conv}(P \cap \mathbb{Z}^2).$$

We want to solve the integer linear program $\max_{x \in P(a,\beta)} c^T x$ for some integral objective vector $c \in \mathbb{Z}^2$. Let x^* denote an optimal integer solution to this ILP and let $x' \in \mathbb{R}^2$ be an optimal solution of the LP relaxation, i. e. the ILP without the integrality constraints. In this exercise, we investigate the strategy of simply rounding x' down componentwise (denoted by $\lfloor x' \rfloor$) to get an integer solution.

a) Is $x' = x^*$ possible? Either give an example or disprove the statement!

b) Is it possible that $x' \neq x^*$, but $\lfloor x' \rfloor = x^*$? Either give an example or disprove the statement!

c) Show that $\lfloor x' \rfloor \in (P(a, \beta))_I$ holds if $a, \beta \geq 0$.

d) Give an example that shows that x^* and $\lfloor x' \rfloor$ need not be "close" (both with respect to Euclidean distance and with respect to the objective value).

Exercise 1.2 (The Assignment Problem)
Let $T = \{t_1, \ldots, t_k\}$ be a finite set of *tasks* that need to be assigned to a set of *workers* $W = \{w_1, \ldots, w_p\}$. Each worker is able to each task, but depending on how well the worker w_i is trained to do a specific task t_j a *cost* c_{ij} is incurred.

a) Assuming $p \geq k$, design an integer linear program that models the assignment of tasks to workers with the objective of minimizing the total cost of the assignment. Each task needs to be assigned to a worker, a task cannot be subdivided and a worker is only able to do one task.

b) In addition to the cost there is now a time requirements T_j for each task t_j and a time budget B_i for each worker w_i. Design an integer linear program that models the assignment of tasks to workers with the objective of minimizing the total cost. Each task needs to be assigned, a task cannot be subdivided. Each worker may receive more than one task, but their total time requirement must not exceed the worker's budget.

c) Assuming $p \geq k$, design an integer linear program that models the assignment of tasks to workers with the objective of minimizing the maximum cost that is incurred among all worker-task-pairs. Each task needs to be assigned to a worker, a task cannot be subdivided and a worker is only able to do one task.

Please turn over.

Exercise 1.3 (The Traveling Salesman Problem)
The traveling salesman problem on the complete undirected graph $G = (V, E)$ on the vertex set $V = [n]$ with edge weights $d : E \to \mathbb{N}_0$ asks for a tour of the nodes ("cities") that visits every node exactly once and has minimum total weight ("tour length"). Consider the following integer linear program:

$$\begin{aligned} \min \sum_{e \in E} d_e x_e & \\ \sum_{e \in \delta(v)} x_e = 2 & \quad \text{for all } v \in V \\ \sum_{e \in E(S)} x_e \leq |S| - 1 & \quad \text{for all } \emptyset \neq S \subsetneq V \qquad (1) \\ x \in \{0,1\}^{|E|} & \end{aligned}$$

a) Give an interpretation of the binary variables and of the constraints in the above program.

b) Consider the program without the *subtour elimination constraints* Equation (1) and show that each feasible solution of that program corresponds to a collection of node disjoint cycles in G.

c) Prove that the integer linear program (with all constraints) correctly models the traveling salesman problem, i. e. there is a bijection between the set of traveling salesman tours on G and the set of feasible integer solutions of the ILP.

d) Show that the subtour elimination constraints may be replaced by the following *cut-set constraints*:

$$\sum_{e \in \delta(S)} x_e \geq 1 \quad \text{for all } \emptyset \neq S \subsetneq V$$

Exercise 1.4
Let P and Q be polytopes in $\mathbb{R}^n$ where $n \geq 1$. Prove or disprove: $(P \setminus Q)_I = (P)_I \setminus (Q)_I$.

Exercise 1.5
Let $P \subset \mathbb{R}^2$ be defined through the following $\mathcal{H}$-presentation:

$$\begin{aligned} -\sqrt{2}x_1 + x_2 &\leq 0 \\ x_1 - \sqrt{2}x_2 &\leq 0 \end{aligned}$$

a) Sketch both P and the integral points contained in P. Take a guess at the integer hull $(P)_I := \text{conv}(P \cap \mathbb{Z}^2)$ based on your sketch!

b) Show that $P = \text{pos}\left\{(1, \sqrt{2})^T, (\sqrt{2}, 1)^T\right\}$ ($\mathcal{V}$-presentation).

c) Prove that $(P)_I = \{0\} \cup \text{int}\,(P)$. You may use the following result without proof: For every line in $\mathbb{R}^2$ with irrational slope there exist integer points arbitrarily close to (and on both sides of) the line.

d) Is $(P)_I$ a polyhedron?

e) Let $Q := \left\{(-\frac{1}{2}, -\frac{1}{2})^T\right\} + P$. Show that the integer hull $(Q)_I$ of Q has an infinite number of vertices.