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Discrete Optimization (MA 3502)

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Exercise Sheet 2

Problem 2.1 (A simple TDI system)
Consider the matrix A and the right hand side vector b given by

$$A := \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad b := \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

a) Show that the system $Ax \leq b$ is not TDI.

b) Now consider the system $A'x \leq b'$ obtained from $Ax \leq b$ by adding the inequality $x_1 \leq 0$. Show that both systems determine the same feasible region.

c) Prove that the system $A'x \leq b'$ is TDI.

Answer to Problem 2.1

a) By definition of TDI, we need to exhibit some integral objective vector $c \in \mathbb{Z}^2$ such that the optimum of max $c^T x$, $Ax \leq b$, is finite but not equal to the dual minimum over all integral dual solutions. Hence we first consider the primal and the dual problem:

$$\begin{aligned} \max\ & c_1x_1 + c_2x_2 \\ & x_1 - x_2 \leq 0 \\ & x_1 + x_2 \leq 0 \end{aligned} \qquad\qquad \begin{aligned} \min\ & 0 \\ & y_1 + y_2 = c_1 \\ & -y_1 + y_2 = c_2 \\ & y \geq 0 \end{aligned}$$

This may seem a little strange at first sight – the dual objective value is always 0, no matter what feasible dual solution y we come up with, and a quick inspection of the primal also yields that 0 is the only vertex. So for every primal objective vector c that yields a finite solution, that solution can only be 0, thus the primal and the dual objective always agree.

So how are we supposed to prove that this is not TDI? There is one catch: Maybe the dual is integer infeasible, so that there is no feasible integer solution that would then yield the objective value of 0 (while there surely are many non-integer feasible solutions to the dual). To see this, remember that the dual asks for a positive combination of the normal vectors of the active constraints at the primal optimal vertex (0 in this case). The normal vectors here are just the rows of A: $(1, -1)^T$ and $(1, 1)^T$. Is there an objective vector that cannot be combined from these using only *integer* coefficients? Let us consider $c = (1, 0)^T$. Then we know $y_1 = y_2$ is necessary to get the second coordinate. However, that means the first coordinate will be $2y_1$, and $2y_1 = 1$ is impossible for y_1 being an integer. This shows that the system is not TDI.

b) By adding the two inequalities of the first system we get $2x_1 \leq 0$. Thus the new inequality is already implied by that system and therefore the feasible region is the same.

c) Adding the new inequality to our system yields the following primal and dual LP:

$$\begin{aligned} \max\ & c_1x_1 + c_2x_2 & \min\ & 0 \\ & x_1 - x_2 \le 0 & & y_1 + y_2 + y_3 = c_1 \\ & x_1 + x_2 \le 0 & & -y_1 + y_2 = c_2 \\ & x_1 \le 0 & & y \ge 0 \end{aligned}$$

This adds a new dual variable y_3 that gives us the necessary flexiblity. For any given integral $c \in \mathbb{Z}^n$ for which the primal optimum is feasible, we know that $c \in \operatorname{pos}\left\{(1,-1)^T, (1,1)^T\right\}$, thus the first coordinate is always nonnegative. We simply set $y_3 := c_1 \ge 0$, and then either $y_2 := c_2$ (if $c_2 \ge 0$) and $y_1 := 0$ or vice versa (if $c_2 < 0$). This gives us a feasible dual integral solution with objective value 0 as desired.

Problem 2.2

For the following matrices, determine whether they are totally unimodular. If the matrices are not totally unimodular, identify corresponding submatrices whose determinant is not in $\{-1, 0, +1\}$.

$$A_1 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ -1 & -1 \\ -1 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Answer to Problem 2.2

A_1 is not totally unimodular since

$$\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$$

is a square submatrix with determinant 2 (take rows 2 and 3, and columns 1, and 3).

A_2 is not totally unimodular since

$$\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

is a square submatrix with determinant -2.

A_3 is unimodular and totally unimodular.

Problem 2.3

Which of the following statements are true and which are false? Give a proof or counterexample.

a) If $A \in \mathbb{Z}^{n\times n}$ is totally unimodular, then all its eigenvalues are in $\{-1, 0, 1\}$

b) If $A \in \mathbb{Z}^{d\times n}$ and $B \in \mathbb{Z}^{m\times n}$ are totally unimodular, then $\begin{pmatrix} A \\ B \end{pmatrix}$ is totally unimodular.

c) If A and B are totally unimodular, then $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is also totally unimodular.

Answer to Problem 2.3

a) False. For instance, $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is totally unimodular and 2 is an eigenvalue.

b) False. For instance, $A = (1,1)$ and $B = (1,-1)$ are totally unimodular, but $\begin{pmatrix} A \\ B \end{pmatrix}$ has determinant 2, and thus not totally unimodular.

c) True. Any square submatrix C has the form $C = \begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix}$, where A' and B' are submatrices of A and B respectively. If these are not square, then C has determinant 0. If they are square, then $\det(C) = \det(A)\det(B) \in \{-1, 0, 1\}$.

Problem 2.4
Let $G = (V, E, c)$ be a weighted, bipartite graph with $V = A \cup B$ and $c : E \to \mathbb{R}_{\geq 0}$. Let $w : V \to \mathbb{R}_{\geq 0}$ be a node weight function and consider the following GENERALIZED STABLE SET problem:

$$\begin{aligned} \max \sum_{x \in V} & w_v x_v \\ & x_a + x_b \leq c_{ab} \quad \text{for all edges } \{a,b\} \in E \\ & x \in \mathbb{R}^{|V|}_{\geq 0} \end{aligned}$$

Show that the system of inequalities given above is totally dual integral.

Answer to Problem 2.4
Let us consider the dual problem of the linear program given in the problem statement. We will use y_{ab}, $\{a,b\} \in E$, as the notation for the dual variables.

$$\begin{aligned} \min \sum_{\{a,b\} \in E} & c_{ab} y_{ab} \\ \sum_{b \in B : \{a,b\} \in E} & y_{ab} \geq w_a \quad \text{for all } a \in A \\ \sum_{a \in A : \{a,b\} \in E} & y_{ab} \geq w_b \quad \text{for all } b \in B \\ & y_{ab} \geq 0 \quad \text{for all } \{a,b\} \in E \end{aligned}$$

The matrix of this problem is the node-edge incidence matrix of the graph G. As G is bipartite, this matrix is totally unimodular (see lecture), therefore the dual has an integral optimal solution whenever the weight vector w is integer. This proves that the system is TDI.

Problem 2.5 (The Matching Polytope)
Let $G = (V, E)$ be a graph on n vertices and m edges. Let S_G denote the node-edge incidence matrix of G and let $\mathcal{M}(G)$ denote the *matching polytope* of G, i. e. the convex hull of all feasible matchings of G:

$$\mathcal{M}(G) := \text{conv}(\{x \in \{0,1\}^m : S_G x \leq \mathbb{1}\})$$

Further, let $P = \{x \in \mathbb{R}^m : S_G x \leq \mathbb{1}, x \geq 0\}$ denote the LP relaxation of $\mathcal{M}(G)$.

a) Determine the dimension $\dim(\mathcal{M}(G))$.

b) Show that all inequalities of the form $x_e \geq 0$ define a facet of $\mathcal{M}(G)$.

c) Show that $P = \mathcal{M}(G)$ if G is a bipartite graph.

Answer to Problem 2.5

a) We get $\dim(\mathcal{M}(G)) = m$ by the following argument: With $0 \in \mathcal{M}(G)$ and $u_e \in \mathcal{M}(G)$ for every $e \in E$ we have $m + 1$ affinely independent vectors contained in $\mathcal{M}(G)$, thus the matching polytope is fully-dimensional.

b) The face $F_e := \{x \in \mathcal{M}(G) : x_e = 0\}$ contains the vector 0 and also $u_{e'}$ for all $e' \in E \setminus \{e\}$, which are all affinely independent. This proves that F_e is a facet for every $e \in E$.

c) For a bipartite graph, the incidence matrix is totally unimodular. Thus P is integral and equal to the matching polytope $\mathcal{M}(G)$.