Technische Universität München, Zentrum Mathematik
Lehrstuhl für Angewandte Geometrie und Diskrete Mathematik

Discrete Optimization (MA 3502)

Prof. Dr. Nicole Megow | Dr. Fidaa Abed | Dr. Michael Ritter

Exercise Sheet 2

Problem 2.1 (A simple TDI system)
Consider the matrix A and the right hand side vector b given by

$$A := \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad b := \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

a) Show that the system $Ax \leq b$ is not TDI.

b) Now consider the system $A'x \leq b'$ obtained from $Ax \leq b$ by adding the inequality $x_1 \leq 0$. Show that both systems determine the same feasible region.

c) Prove that the system $A'x \leq b'$ is TDI.

Problem 2.2
For the following matrices, determine whether they are totally unimodular. If the matrices are not totally unimodular, identify corresponding submatrices whose determinant is not in $\{-1, 0, +1\}$.

$$A_1 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ -1 & -1 \\ -1 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Problem 2.3
Which of the following statements are true and which are false? Give a proof or counterexample.

a) If $A \in \mathbb{Z}^{n \times n}$ is totally unimodular, then all its eigenvalues are in $\{-1, 0, 1\}$

b) If $A \in \mathbb{Z}^{d \times n}$ and $B \in \mathbb{Z}^{m \times n}$ are totally unimodular, then $\begin{pmatrix} A \\ B \end{pmatrix}$ is totally unimodular.

c) If A and B are totally unimodular, then $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is also totally unimodular.

Problem 2.4
Let $G = (V, E, c)$ be a weighted, bipartite graph with $V = A \cup B$ and $c : E \to \mathbb{R}_{\geq 0}$. Let $w : V \to \mathbb{R}_{\geq 0}$ be a node weight function and consider the following Generalized Stable Set problem:

$$\begin{aligned} \max \sum_{x \in V} & w_v x_v \\ x_a + x_b &\leq c_{ab} \quad \text{for all edges } \{a, b\} \in E \\ x &\in \mathbb{R}_{\geq 0}^{|V|} \end{aligned}$$

Show that the system of inequalities given above is totally dual integral.

Please turn over.

Problem 2.5 (The Matching Polytope)
Let $G = (V, E)$ be a graph on n vertices and m edges. Let S_G denote the node-edge incidence matrix of G and let $\mathcal{M}(G)$ denote the *matching polytope* of G, i. e. the convex hull of all feasible matchings of G:

$$\mathcal{M}(G) := \operatorname{conv}(\{x \in \{0,1\}^m : S_G x \leq \mathbb{1}\})$$

Further, let $P = \{x \in \mathbb{R}^m : S_G x \leq \mathbb{1}, x \geq 0\}$ denote the LP relaxation of $\mathcal{M}(G)$.

a) Determine the dimension $\dim(\mathcal{M}(G))$.

b) Show that all inequalities of the form $x_e \geq 0$ define a facet of $\mathcal{M}(G)$.

c) Show that $P = \mathcal{M}(G)$ if G is a bipartite graph.