Discrete Optimization (MA 3502)

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Exercise Sheet 3

Problem 3.1

Recall from the lecture:

Let $A = (a_{ij}) \in \{-1, 0, +1\}^{m \times n}$ be such that each column of A contains at most two nonzero entries. Then A is totally unimodular if and only inf there exists a partition (I_1, I_2) of the row indices $[m]$ with the property

$$a_{i_1,j} \cdot a_{i_2,j} < 0 \quad \Leftrightarrow \quad (\{i_1, i_2\} \subset I_1 \text{ or } \{i_1, i_2 \subset I_2\})$$

for all $i_1, i_2 \in [m]$ with $i_1 \neq i_2$ and $a_{i_1,j}, a_{i_2,j} \neq 0$.

Describe of a polynomial time algorithm which tests whether a given matrix A fulfills the conditions of the previous Corollary.

Hint: Try to construct a graph that is bipartite if and only if the conditions of the corollary are met.

Answer to Problem 3.1

Clearly, it can checked in polynomial time whether the columns of A have at most two non-zero entries. Thus assuming this to be true, we construct a graph G which is is bipartite if and only if the partition as described in Corollary 3.3.3 is possible. The idea is to construct a so-called *conflict graph*.

We assign one vertex to each row of A. If a column of A contains two 1 entries or two -1 entries we connect the vertices of the corresponding rows with an edge, thus forcing those row nodes to be in different parts of the partition if the graph is bipartite. If a column contains one 1 and one -1 entry we add an additional vertex to the graph and connect this new vertex with the vertices corresponding to the -1 and 1 entries, thus forcing the row nodes to be in the same part of the partition for a bipartite graph. (Alternatively, the two vertices could be identified as being one vertex.)

Obviously, every partition of the rows of A as in Corollary 3.3.3 leads to a partition of the vertices in the constructed graph which shows that it is bipartite and vice versa. Bipartiteness of a graph can be checked in polynomial time: For instance, apply breadth-first-search and color the layers alternatingly; the graph is not bipartite if and only if a conflict arises.

Problem 3.2

Let $A \in \{-1, 0, +1\}^{m \times n}$. Show that if A is totally unimodular then the following holds:

a) Every regular submatrix of A has at least one row with an odd number of non-zero entries.

b) The sum over all entries in every square submatrix with even row and column sums is divisible by 4.

Answer to Problem 3.2

Every $k \times k$-submatrix B of a totally unimodular matrix A is (by definition) again totally unimodular. Thus, by Theorem 3.3.2, there is for every $k \times k$-submatrix B of A a partition (I_1, I_2) of the columns b_i of B such that $\sum_{i \in I_1} b_i - \sum_{i \in I_2} b_i \in \{0, \pm 1\}^k$; in other words there exists an $x \in \{-1, +1\}^k$ such that $Bx = \{0, \pm 1\}^k$.

a) If every row of B contained an even number of non-zero entries, then Bx would contain only even entries. Since $Bx \in \{-1, 0, 1\}^k$ this would imply $Bx = 0$ and thus $x = 0$. However, this contradicts $x \in \{-1, +1\}^k$.

b) Consider a (square) submatrix B with row indices I and column indices J, and consider the (by Theorem 3.3.2) claimed partition (I_1, I_2) of I such that $\sum_{i \in I_1} b_i - \sum_{i \in I_2} b_i \in \{0, \pm 1\}^k$. In fact, the stronger property $\sum_{i \in I_1} b_i - \sum_{i \in I_2} b_i = 0$ holds, because the column sums of B yield even numbers. We therefore have

$$\sum_{i \in I_1} \sum_{j \in J} b_{ij} = \sum_{i \in I_2} \sum_{j \in J} b_{ij} =: 2s$$
$$\text{and thus} \quad \sum_{i \in I} \sum_{j \in J} b_{ij} = \sum_{i \in I_1} \sum_{j \in J} b_{ij} + \sum_{i \in I_2} \sum_{j \in J} b_{ij} = 2s + 2s = 4s,$$

proving the claim.

Remark: That B should be a square submatrix is not really needed in (b), but it is stated since it can be computationally easier to check (than checking all submatrices).

Problem 3.3

Given a digraph $G = (V, A)$, two distinguished nodes $s, t \in V$ with $(t, s) \in A$, nonnegative capacities c_{ij} for $(i, j) \in A$ and $c_{ts} = \infty$, find a maximum flow from s to t. This is known as *Max Flow problem*, which can be formulated as LP (see (a)) where at each node we have flow conservation (*flow in=flow out*).

In this exercise we prove the celebrated max flow-min cut theorem of Ford and Fulkerson. For this we need the notion of a cut: for any $U \subset V$ the *cut* $\delta^+(U)$ is defined as $\delta^+(U) := \{(i, j) : i \in U, j \notin U, (i, j) \in A\}$; its capacity is defined as $\sum_{(i,j) \in \delta^+(U)} c_{ij}$.

Theorem (Ford and Fulkerson 1956): The maximum amount of flow from s to t equals the minimum capacity of a cut $\delta^+(U)$ where $s \in U$ and $t \notin U$. Furthermore, if the c_{ij} are integral then there is an integral flow that achieves the maximum flow.

a) We can formulate the Maximum Flow problem as LP:

$$\text{(MF)} \qquad \begin{array}{lll} \max & x_{ts} & \\ \text{s.t.} & \sum_{k : (i,k) \in A} x_{ik} - \sum_{k : (k,i) \in A} x_{ki} = 0, & \forall i \in V, \\ & 0 \le x_{ij} \le c_{ij}, & \forall (i, j) \in A. \end{array}$$

Prove: *If the capacities c_{ij} are integral then every extreme point of the feasible region of MF is integral, and so there is a maximum flow that is integral.*

b) Verify that the dual of (MF) is

$$\text{(MC)} \qquad \begin{array}{lll} \min & \sum_{(i,j) \in A} c_{ij} w_{ij} & \\ \text{s.t.} & u_i - u_j + w_{ij} \ge 0, & \forall (i, j) \in A \setminus \{(t, s)\}, \\ & u_t - u_s \ge 1, & \\ & w_{ij} \ge 0, & \forall (i, j) \in A. \end{array}$$

c) Show that every extreme point solution (u, w) of (MC) is integral and we can assume $u_s = 0$.

d) Given a solution as in c), let $X := \{j \in V : u_j \le 0\}$ and $X' := V \setminus X = \{j \in V : u_j \ge 1\}$. Show that

$$\sum_{(i,j) \in A} c_{ij} w_{ij} \ge \sum_{(i,j) \in \delta^+(X)} c_{ij} w_{ij} \ge \sum_{(i,j) \in \delta^+(X)} c_{ij},$$

where the lower bound is attainable for a $0 - 1$ solution of MC.

e) Based on d) conclude the proof of the max flow-min cut theorem.

Answer to Problem 3.3

a) The feasible region of MF is non-empty ($x = 0$ is feasible since $c_{ij} \geq 0$, $\forall (i,j) \in A$) and bounded (since $0 \leq x_{ij} \leq c_{ij}, \forall (i,j) \in A \setminus \{(t,s)\}$). There is thus a finite solution attained at a vertex of the feasible region, and since the feasible region of MF can be written as

$$\left\{ x \in \mathbb{R}^{|A|} : \begin{pmatrix} -S_G \\ S_G \\ I' \end{pmatrix} x \leq \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}, \, x \geq 0 \right\}, \tag{1}$$

where I' is the $|A| \times |A|$ identity matrix except that the column corresponding to the (t,s) arc is the zero column. Since $\begin{pmatrix} -S_G \\ S_G \\ I \end{pmatrix}$ is totally unimodular (see Prop. 3.4.1 and Rem. 3.3.1) we also have totally unimodularity of $\begin{pmatrix} -S_G \\ S_G \\ I' \end{pmatrix}$, and thus we have proved the assertion (see Cor. 3.2.13).

b) Since the dual of $\{\min c^T x, \text{ s.t. } Ax \leq b\}$ is $\{\max b^T y, \text{ s.t. } A^T y = c, y \geq 0\}$ (see also Exercise 1.2) we obtain the dual of MF using Equation (1) (the corresponding A^T, except for the (t,s)-row is here $(-S_G^T \;\; S_G^T \;\; I \;\; -I)$) as

$$\text{(MC')} \qquad \begin{array}{lll} \min & \sum_{(i,j)\in A} c_{ij} w_{ij} & \\ \text{s.t.} & (u''_i - u'_i) - (u''_j - u'_j) + w_{ij} - w'_{ij} = 0, & \forall (i,j) \in A \setminus \{(t,s)\}, \\ & (u''_t - u'_t) - (u''_s - u'_s) - w'_{ts} = 1, & \\ & u'_i, u''_i \in V, & \forall i \in V, \\ & w_{ij}, \, w'_{ij} \geq 0, & \forall (i,j) \in A. \end{array}$$

Note that there is no w_{ts} in $(u''_t - u'_t) - (u''_s - u'_s) - w'_{ts} = 1$, since we assume no capacity constraint on the extra (t,s) arc.

Setting $u_i := u''_i - u'_i$ (which has now no sign restriction anymore) and considering w'_{ij} as slack variable, MC' simplifies to

$$\text{(MC)} \qquad \begin{array}{lll} \min & \sum_{(i,j)\in A} c_{ij} w_{ij} & \\ \text{s.t.} & u_i - u_j + w_{ij} \geq 0, & \forall (i,j) \in A \setminus \{(t,s)\}, \\ & u_t - u_s \geq 1, & \\ & w_{ij} \geq 0, & \forall (i,j) \in A. \end{array}$$

c) MC can be written as

$$\left\{ \min \sum_{(i,j)\in A} c_{ij} w_{ij}, \text{ s.t. } \begin{pmatrix} -S_G^T & S_G^T & I \end{pmatrix} \begin{pmatrix} u' \\ u'' \\ w \end{pmatrix} \geq b, \quad w_{ts} = 0, u', \, u'', \, w \geq 0 \right\},$$

where b is the zero vector except that the component corresponding to the (t,s) arc contains a 1.

The matrix S_G is totally unimodular (Prop. 3.4.1) and so is $(-S_G^T \;\; S_G^T \;\; I)$; see Rem. 3.3.1. The extreme points are thus integral (note that the $w_{ts} \leq 0$ constraint can be added as particular row of an identity matrix placed under the $(-S_G^T \quad S_G^T \quad I)$ matrix; this is thus again totally unimodular).

We can assume that $u_s = 0$, because we can replace u_j by $u_j + \alpha$ for all $j \in V$; this replacement gives still a feasible u and the optimal value does not change.

d) The first inequality is clear, because the sum on the left contains possibly more terms (and they are non-negative). The last inequality follows, because $w_{ij} \geq u_j - u_i \geq 1$ for $(i,j) \in A$ with $i \in X$ and $j \in X'$.

It remains to be shown that there is a $0-1$ solution of MC that attains the bound $\sum_{(i,j)\in\delta^+(X)} c_{ij}$ where X is given. We set $\tilde{y}_i := 0$ if $i \in X$, $\tilde{y}_i := 1$ if $i \in X'$, $\tilde{w}_{ij} = 1$ if $(i,j) \in A$ with $i \in X$, $j \in X'$; otherwise we set $\tilde{w}_{ij} = 0$. It is easily checked that $(\tilde{y}, \tilde{w})$ is feasible for MC, and clearly the corresponding objective function value is $\sum_{(i,j)\in\delta^+(X)} c_{ij}$.

e) For the solution from d) that attains the lower bound, we have $s \in X$ and $t \in X'$. The set $\{(i,j) : w_{ij} = 1\}$ is the set of arcs of the cut $\delta^+(X)$. We have seen that the optimal objective function value is $\sum_{(i,j)\in\delta^+(X)} c_{ij}$; this is by definition the capacity of the cut $\delta^+(X)$. And as the optimal value of MC is equal to the optimal value of MF (strong duality) we thus conclude that the minimum capacity of a cut $\delta^+(U)$ where $s \in U$ and $t \notin U$ is equal to the maximal flow from s to t.

Problem 3.4

Show that $A \in \{-1,0,1\}^{m\times n}$ is totally unimodular if and only if the polytope

$$P := \{x \in \mathbb{R}^n : a \leq Ax \leq b, c \leq x \leq d\}$$

is integral for all $a, b \in \mathbb{Z}^m$ and $c, d \in \mathbb{Z}^n$.

Answer to Problem 3.4

We start by substituting $y := x - c$, i. e. $x = y + c$, the polytope then becomes

$$P' := \{y \in \mathbb{R}^n : a - Ac \leq Ay \leq b - Ac, 0 \leq y \leq d - c\}.$$

Of course, P' is integral if and only if P is, because both A and c are integral, thus we only need to show integrality of P'. For the sake of easier notation, we will introduce the abbreviations $a' := a - Ac$, $b' := b - Ac$ and $d' := d - c$, so that $P' = \{y \in \mathbb{R}^n : a' \leq Ay \leq b', 0 \leq y \leq d'\}$.

The system $a' \leq Ay \leq b'$, $y \leq d'$ can be expressed as

$$\left\{ y \in \mathbb{R}^n : \begin{pmatrix} A \\ -A \\ I_n \end{pmatrix} \leq \begin{pmatrix} b' \\ -a' \\ d \end{pmatrix}, \right\}.$$

The claim now follows from the following lemma and the theorem in the lecture.

Lemma 1

If A is a totally unimodular matrix, then so are $(A \quad -A)$ and $(A \quad I)$.

To show the lemma, let A be some totally unimodular matrix and consider a submatrix of $(A \quad -A)$. If determinant of that submatrix is 0, nothing is left to prove. Otherwise, the submatrix contains some columns from A and some from $-A$. By multiplying the latter with -1, we get a submatrix of A while the determinant has not changed its absolute value (possibly its sign). By total unimodularity of A, the determinant of that transformed submatrix is 0, -1 or $+1$, thus the determinant of the original matrix is also 0, -1 or $+1$, which shows that $(A \quad -A)$ is totally unimodular.

For $(A \quad I)$, we first consider the case $(A \quad u_i)$ where u_i is some unit vector. Consider some submatrix of this matrix, then that is either just a submatrix of A (in which case its determinant is in $\{-1, 0, +1\}$ by total unimodularity of A), it contains a 0 column (i. e. the determinant is 0) or it contains a unit vector as a column. In the latter case, we expand the determinant along that column and get ± 1 times the determinant of some submatrix of A, which is again in $\{-1, 0, 1\}$. Inductive application of this argument finally shows the claim.