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Discrete Optimization (MA 3502)

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Exercise Sheet 3

Problem 3.1

Recall from the lecture:

Let $A = (a_{ij}) \in \{-1, 0, +1\}^{m\times n}$ be such that each column of A contains at most two nonzero entries. Then A is totally unimodular if and only inf there exists a partition (I_1, I_2) of the row indices $[m]$ with the property

$$a_{i_1,j} \cdot a_{i_2,j} < 0 \quad \Leftrightarrow \quad (\{i_1, i_2\} \subset I_1 \text{ or } \{i_1, i_2 \subset I_2\})$$

for all $i_1, i_2 \in [m]$ with $i_1 \neq i_2$ and $a_{i_1,j}, a_{i_2,j} \neq 0$.

Describe of a polynomial time algorithm which tests whether a given matrix A fulfills the conditions of the previous Corollary.

Hint: Try to construct a graph that is bipartite if and only if the conditions of the corollary are met.

Problem 3.2

Let $A \in \{-1, 0, +1\}^{m\times n}$. Show that if A is totally unimodular then the following holds:

a) Every regular submatrix of A has at least one row with an odd number of non-zero entries.

b) The sum over all entries in every square submatrix with even row and column sums is divisible by 4.

Problem 3.3

Given a digraph $G = (V, A)$, two distinguished nodes $s, t \in V$ with $(t, s) \in A$, nonnegative capacities c_{ij} for $(i, j) \in A$ and $c_{ts} = \infty$, find a maximum flow from s to t. This is known as *Max Flow problem*, which can be formulated as LP (see (a)) where at each node we have flow conservation (*flow in=flow out*).
In this exercise we prove the celebrated max flow-min cut theorem of Ford and Fulkerson. For this we need the notion of a cut: for any $U \subset V$ the *cut* $\delta^+(U)$ is defined as $\delta^+(U) := \{(i, j) : i \in U, j \notin U, (i, j) \in A\}$; its capacity is defined as $\sum_{(i,j)\in\delta^+(U)} c_{ij}$.

Theorem (Ford and Fulkerson 1956): The maximum amount of flow from s to t equals the minimum capacity of a cut $\delta^+(U)$ where $s \in U$ and $t \notin U$. Furthermore, if the c_{ij} are integral then there is an integral flow that achieves the maximum flow.

a) We can formulate the Maximum Flow problem as LP:

$$\text{(MF)} \qquad \begin{array}{lll} \max & x_{ts} & \\ \text{s.t.} & \sum_{k:(i,k)\in A} x_{ik} - \sum_{k:(k,i)\in A} x_{ki} = 0, & \forall i \in V, \\ & 0 \le x_{ij} \le c_{ij}, & \forall (i,j) \in A. \end{array}$$

Prove: *If the capacities c_{ij} are integral then every extreme point of the feasible region of MF is integral, and so there is a maximum flow that is integral.*

Please turn over.

b) Verify that the dual of (MF) is

$$
\text{(MC)} \quad \begin{array}{lll} \min & \sum_{(i,j)\in A} c_{ij} w_{ij} & \\ \text{s.t.} & u_i - u_j + w_{ij} \geq 0, & \forall (i,j) \in A \setminus \{(t,s)\}, \\ & u_t - u_s \geq 1, & \\ & w_{ij} \geq 0, & \forall (i,j) \in A. \end{array}
$$

c) Show that every extreme point solution (u, w) of (MC) is integral and we can assume $u_s = 0$.

d) Given a solution as in c), let $X := \{j \in V \ : \ u_j \leq 0\}$ and $X' := V \setminus X = \{j \in V \ : \ u_j \geq 1\}$. Show that

$$
\sum_{(i,j)\in A} c_{ij} w_{ij} \geq \sum_{(i,j)\in \delta^+(X)} c_{ij} w_{ij} \geq \sum_{(i,j)\in \delta^+(X)} c_{ij},
$$

where the lower bound is attainable for a $0-1$ solution of MC.

e) Based on d) conclude the proof of the max flow-min cut theorem.

Problem 3.4

Show that $A \in \{-1, 0, 1\}^{m\times n}$ is totally unimodular if and only if the polytope

$$
P := \{x \in \mathbb{R}^n : a \leq Ax \leq b, c \leq x \leq d\}
$$

is integral for all $a, b \in \mathbb{Z}^m$ and $c, d \in \mathbb{Z}^n$.