Discrete Optimization (MA 3502)

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Exercise Sheet 4

***Problem 4.1** (Knapsack Problem)
The 0-1-KNAPSACK-problem can be defined as follows:

GIVEN: n goods with weights $a_1, \ldots, a_n \in \mathbb{N}$, values $c_1, \ldots, c_n \in \mathbb{N}$, and a weight-limit $b \in \mathbb{N}$. All goods are available only once.
TASK: Find a collection of goods not exceeding the weight-limit b, of maximal value.

(Without loss of generality (Wlog) one may assume $b \geq \max_j a_j$.)

For $r \in [n] := \{1, \ldots, n\}$ and $b' \in [b]$ define

$$z_r(b') := \max \left\{ \sum_{i=1}^{r} c_i x_i \mid \sum_{i=1}^{r} a_i x_i \leq b' \ , \ x_i \in \{0,1\} \ i \in [r] \right\}$$

Hence, our goal is to compute $z_n(b)$.

a) Show $z_n(b) = \max\{z_{n-1}(b), z_{n-1}(b - a_n) + c_n\}$.

b) Use (a) to construct an algorithms solving 0-1-KNAPSACK in time $O(b \cdot n)$.

c) Discuss the running time of the algorithm with respect to the definition of efficiency in complexity theory.

Problem 4.2 ($\mathcal{NP}$-hardness of the knapsack problem)
Show that the knapsack problem is $\mathcal{NP}$-hard by giving a reduction of the $\mathcal{NP}$-hard *partition problem.*

Problem: Partition
Input: Set of natural numbers $A = \{a_1, \ldots, a_n\} \subset \mathbb{N}$
Question: Is there a partition $A_1, A_2 := A/A_1$ of A such that

$$\sum_{a \in A_1} a = \sum_{a \in A_2} a?$$

to the knapsack problem.

Problem 4.3 (Dynamic Programming for TSP)
Design an algorithm that solves the metric TSP in running time $\mathcal{O}\left(n^2 2^n\right)$ using a dynamic programming approach. Use the following notation and variables:

- $S \subset \{2, 3, \ldots, n\}$ and $s \in S$.
- $\text{bestpath}[S, s] :=$ shortest path from 1 to s which passes through all vertices in $S \cup \{1\}$.

Please turn over.

- $\text{cost}[S, s] :=$ length of $\text{bestpath}[S, s]$.

Where in your algorithm / proof do you use the fact that the distance function is metric?

***Problem 4.4** (The Chvátal closure)
Given the following polyhedron P.

$$-2x_1 + x_2 \leq 0$$
$$2x_1 + x_2 \leq 6$$
$$-x_2 \leq -1$$

a) Show that the convex hull conv{(1,1),(2,1),(1,2),(2,2),(1.5,2.5)} is the (first) *Chvátal closure* of the P.

b) Prove that the inequality $x_2 \leq 2$ can not be derived from the inequalities describing P.

c) Show that P has Chvatal rank 2.