Discrete Optimization (MA 3502)

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Exercise Sheet 5

Problem 5.1

Use the Gomory cutting plane algorithm to compute an optimal solution of the follwing ILP. Draw a sketch illustrating the situation in each step.

$$
\begin{aligned}
\max\ & 5x_1 + 3x_2 \\
& 2x_1 + 3x_2 \leq 10, \\
& x_1 - 2x_2 \leq 0 \\
& x \in \mathbb{Z}^2
\end{aligned}
$$

Answer to Problem 5.1

Let us start by sketching the feasible region of the LP relaxation in Figure 1.

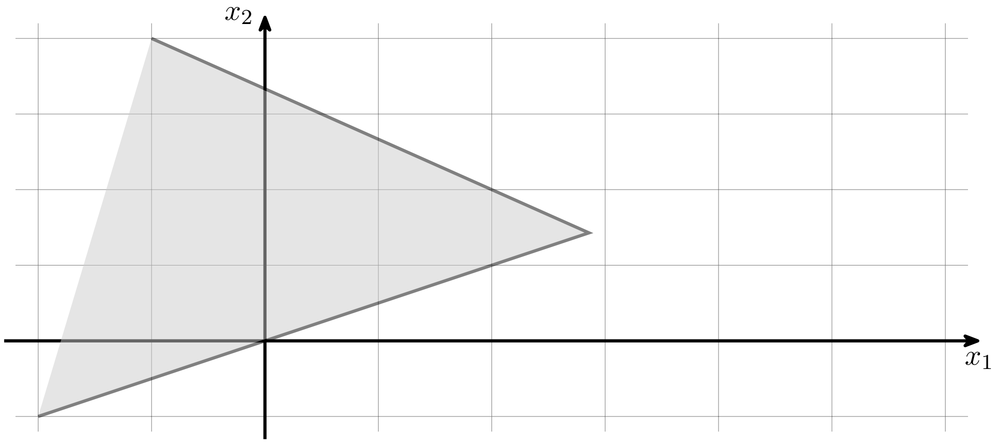

Figure 1: Feasible region of the LP.

For the Gomory cut computations, we first to transform the problem into canonical form (i. e. $\max c^T x$, $Ax = b$, $x \geq 0$), hence we split the unrestricted variables into "+"- and "−"-parts and add slack variables, thus obtaining the following problem formulation:

$$
\begin{aligned}
\max\ & 5x_1^+ - 5x_1^- + 3x_2^+ - 3x_2^- \\
& 2x_1^+ - 2x_1^- + 3x_2^+ - 3x_2^- + x_3 = 10, \\
& x_1^+ - x_1^- - 2x_2^+ + 2x_2^- + x_4 = 0 \\
& x \in \mathbb{N}_0^6
\end{aligned}
$$

By setting $x_3 = 10, x_4 = 0$ as the basis variables (and all other, non-basis variables, to 0) we obtain a feasible solution and an initial basis for our first tableau. Recall the general form of the simplex tableau:

$c^T x$	$c_B^T A_B^{-1} A - c$
$A_B^{-1} b$	$A_B^{-1} A$

Inserting our LP yields the simplex tableau:

	x_1^+	x_1^-	x_2^+	x_2^-	$\mathbf{x_3}$	$\mathbf{x_4}$
0	-5	5	-3	3	0	0
10	2	-2	3	-3	1	0
0	1	-1	-2	2	0	1

We start by selecting the pivot element's column: Any negative reduced cost column will do, we usually select the one with the most negative entry, thus x_1^+ will enter the basis. To obtain the leaving variable, we consider the quotients of leftmost entry and entry in the x_1^+-column and choose the element with minimum ratio (among all positive entries in the column). Here we have the ratios $\frac{10}{2} = 5$ and $\frac{0}{1} = 0$, thus our pivot element is in the last row of the tableau and x_4 will leave the basis. The 0-entry in the left column of the tableau already tells us that this step will not increase the objective function, geometrically we are changing the basis but staying at the same vertex of the polytope. The simplex tableau becomes:

	$\mathbf{x_1^+}$	x_1^-	x_2^+	x_2^-	$\mathbf{x_3}$	x_4
0	0	0	-13	13	0	5
10	0	0	7	-7	1	-2
0	1	-1	-2	2	0	1

As the reduced cost row still contains negative entries, we have not yet reached an optimum, thus we select x_2^+ as the next pivot column. There is only one positive entry in the tableau in that column, so our pivot element is in position $(1, 3)$ (the entry 7), meaning that x_2^+ will enter the basis and x_3 will leave. After that pivot step, we obtain the following tableau:

	$\mathbf{x_1^+}$	x_1^-	$\mathbf{x_2^+}$	x_2^-	x_3	x_4
$130/7$	0	0	0	0	$13/7$	$9/7$
$10/7$	0	0	1	-1	$1/7$	$-2/7$
$20/7$	1	-1	0	0	$2/7$	$3/7$

The solution $(\frac{20}{7}, 0, \frac{10}{7}, 0, 0, 0)$ is now LP-optimal but still fractional. Hence we have to add a Gomory cut, and will start by cutting with respect to the x_1^+ variable. The cut is therefore obtained from the last row by rounding down the coefficients, we get

$$x_1^+ - x_1^- \leq \left\lfloor \frac{20}{7} \right\rfloor = 2, \quad \text{corresponding to } x_1 \leq 2 \text{ in our original model (see Figure 2).}$$

To get this new inequality into the simplex tableau, we rewrite it using a new slack variable x_5 and get $x_1^+ - x_1^- + x_5 = 2$, hence our problem now changes to

$$\begin{aligned} \max\ & 5x_1^+ - 5x_1^- + 3x_2^+ - 3x_2^- \\ & 2x_1^+ - 2x_1^- + 3x_2^+ - 3x_2^- + x_3 = 10, \\ & x_1^+ - x_1^- - 2x_2^+ + 2x_2^- + x_4 = 0 \\ & x_1^+ - x_1^- + x_5 = 2 \\ & x \in \mathbb{N}_0^6 \end{aligned}$$

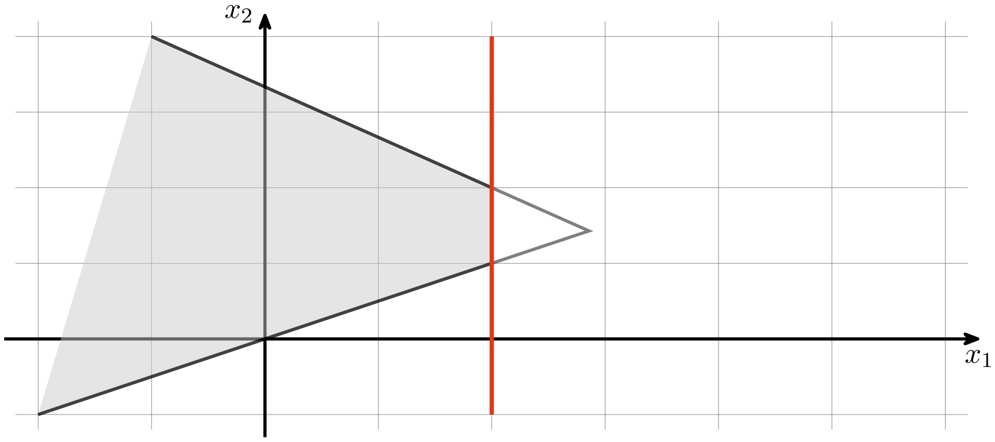

Figure 2: First Gomory cut $x_1 \leq 2$.

We restart the simplex algorithm, using again the slack variables as basic variables to get an initial solution:

	x_1^+	x_1^-	x_2^+	x_2^-	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_5}$
0	−5	5	−3	3	0	0	0
10	2	−2	3	−3	1	0	0
0	1	−1	−2	2	0	1	0
2	1	−1	0	0	0	0	1

Just to see a different computation, let us use x_2^+ as the first pivot column, then our pivot element is in position $(1, 3)$ and x_3 will leave the basis. This yields the following new simplex tableau:

	x_1^+	x_1^-	$\mathbf{x_2^+}$	x_2^-	x_3	$\mathbf{x_4}$	$\mathbf{x_5}$
10	−3	3	0	0	1	0	0
10/3	2/3	−2/3	1	−1	1/3	0	0
20/3	7/3	−7/3	0	0	2/3	1	0
2	1	−1	0	0	0	0	1

The next step will pivot x_1^+ into the basis and x_5 out of the basis, thus we get this new simplex tableau:

	$\mathbf{x_1^+}$	x_1^-	$\mathbf{x_2^+}$	x_2^-	x_3	$\mathbf{x_4}$	x_5
16	0	0	0	0	1	0	3
2	0	0	1	−1	1/3	0	−2/3
2	0	0	0	0	2/3	1	−7/3
2	1	−1	0	0	0	0	1

This tableau yields an LP-optimal solution of $(2, 0, 2, 0, 0, 2, 0)$ which is integer. Remember that we would usually not care about non-integralities in the slack variables, although in this case even the slacks are integer. Our optimal solution (transformed back to the original problem formulation) is therefore $x^* = (2, 2)$.

Problem 5.2

Consider the polyhedron $P = \{x \in \mathbb{R}^2 : Ax \le b\}$ given by

$$A := \begin{pmatrix} -3 & -4 \\ -1 & 1 \\ 4 & 6 \\ 4 & -10 \end{pmatrix}, \quad b := \begin{pmatrix} -8 \\ 2 \\ 27 \\ 3 \end{pmatrix}.$$

A $\mathcal{V}$-presentation of P is given by

$$P = \operatorname{conv}\left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} \frac{3}{2} \\ \frac{7}{2} \end{pmatrix}, \begin{pmatrix} \frac{9}{2} \\ \frac{3}{2} \end{pmatrix}, \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix} \right\}.$$

a) Draw a sketch of both P and its integer hull P_I.

b) Can you determine some proper cut for P_I just by "staring" at the inequalities?

c) Consider the vertex $x^* = (3/2, 7/2)^T$ of P. Compute a Gomory cut at x^* with respect to the first component.

d) Using $B = \{2, 3\}$ as a starting basis corresponding to x^*, use the Gomory cutting plane algorithm to compute the optimal solution to the following ILP:

$$\begin{aligned} \max\ & x_2 \\ & Ax \le b \\ & x \in \mathbb{Z}^2 \end{aligned}$$

Answer to Problem 5.2

a) Figure 3 shows a sketch of both P and its integer hull.

b) The last two inequalities have only even coefficients on the left hand side, but an odd right hand side. Thus, for each integral vector x, the left hand side of both inequalities will be even, and hence the right hand side may be rounded down to the nearest even integer. Formally, we can divide both inequalities by 2 and then apply rounding. This procedure yields the cuts

$$\begin{aligned} 2x_1 + 3x_2 &\le \left\lfloor \frac{27}{2} \right\rfloor = 13 \\ \text{and} \quad 2x_1 - 5x_2 &\le \left\lfloor \frac{3}{2} \right\rfloor = 1. \end{aligned}$$

Both cuts are illustrated in red in Figure 3.

c) Substituting x^* into the inequalities yields $Ax^* = (-37/2, 2, 27, -29)^T$, thus the second and third inequality are active in x^*. For doing Gomory cuts, we first transform the problem into canonical form. We add the objective max x_2 from the last part of this problem right away, as we will later need that anyway.

$$\begin{aligned} \max\ & x_2^+ - x_2^- \\ & -3x_1^+ + 3x_1^- - 4x_2^+ + 4x_2^- + x_3 = -8 \\ & -x_1^+ + x_1^- + x_2^+ - x_2^- + x_4 = 2 \\ & 4x_1^+ - 4x_1^- + 6x_2^+ - 6x_2^- + x_5 = 27 \\ & 4x_1^+ - 4x_1^- - 10x_2^+ + 10x_2^- + x_6 = 3 \\ & x \ge 0 \end{aligned}$$

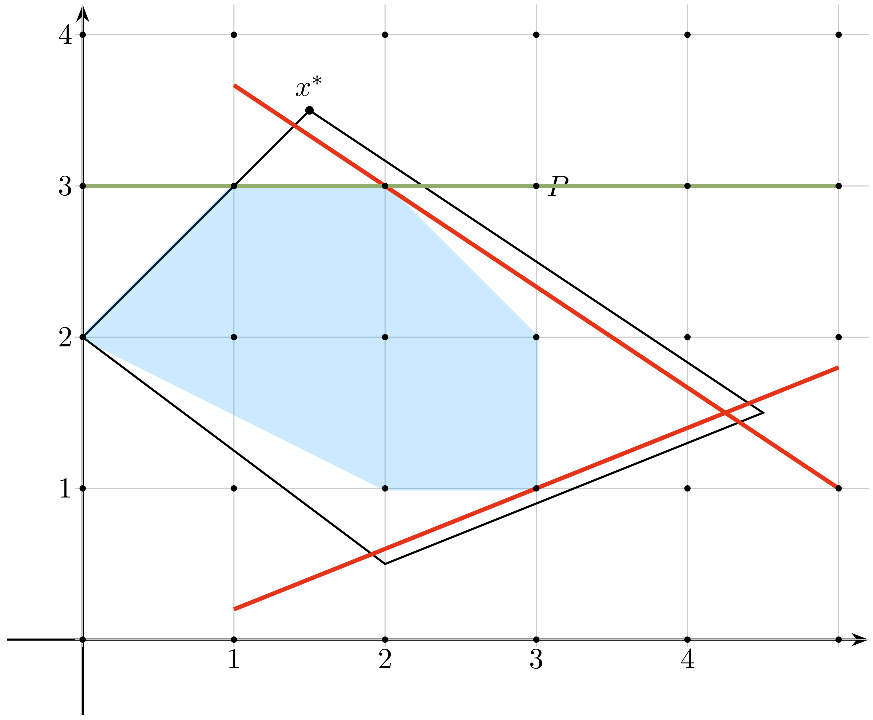

Figure 3: The polytope P, its integer hull and two cuts derived by "staring".

The basic variables corresponding to $x^* = (3/2, 7/2)^T$ are then $x_1^+ = 3/2$, $x_2^+ = 7/2$, $x_3 = 21/2$ and $x_6 = 32$, thus we need to compute the simplex tableau corresponding to this basis. We start with

$$A_B = \begin{pmatrix} -3 & -4 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 4 & 6 & 0 & 0 \\ 4 & -10 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad A_B^{-1} = \begin{pmatrix} 0 & -6/10 & 1/10 & 0 \\ 0 & 4/10 & 1/10 & 0 \\ 1 & -2/10 & 7/10 & 0 \\ 0 & 64/10 & 6/10 & 1 \end{pmatrix}$$

This gives us the simplex tableau corresponding to the given vertex:

	$\mathbf{x_1^+}$	x_1^-	$\mathbf{x_2^+}$	x_2^-	$\mathbf{x_3}$	x_4	x_5	$\mathbf{x_6}$
$7/2$	0	0	0	0	0	$4/10$	$1/10$	0
$3/2$	1	-1	0	0	0	$-6/10$	$1/10$	0
$7/2$	0	0	1	-1	0	$4/10$	$1/10$	0
$21/2$	0	0	0	0	1	$-2/10$	$7/10$	0
32	0	0	0	0	0	$64/10$	$6/10$	1

A cut to the first component corresponds to a cut with respect to the first row of this simplex tableau here, the cut inequality obtained from that row is

$$x_1^+ - x_1^- - x_4 \leq 1, \quad \text{or (with new slack variable)} \quad x_1^+ - x_1^- - x_4 + x_7 = 1.$$

d) We simply continue with the computations above. Looking at the last simplex tableau, we see that our solution already was LP optimal, thus we use the obtained Gomory cut and integrate it into our tableau, introducing a new slack variable. We will also turn x_3 into two different slacks x_3^+ and x_3^- to be able to get a feasible solution without any computational effort. The model then looks like this:

$$
\begin{aligned}
\max\ & x_2^+ - x_2^- \\
& -3x_1^+ + 3x_1^- - 4x_2^+ + 4x_2^- + x_3^+ - x_3^- = -8 \\
& -x_1^+ + x_1^- + x_2^+ - x_2^- + x_4 = 2 \\
& 4x_1^+ - 4x_1^- + 6x_2^+ - 6x_2^- + x_5 = 27 \\
& 4x_1^+ - 4x_1^- - 10x_2^+ + 10x_2^- + x_6 = 3 \\
& x_1^+ - x_1^- - x_4 + x_7 = 1 \\
& x \geq 0
\end{aligned}
$$

A basic feasible solution is then $x_3^- = 8$, $x_4 = 2$, $x_5 = 27$, $x_6 = 3$, $x_7 = 3$ (because we already set x_4 which also appears in the last equality). This yields

$$
A_B = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad A_B^{-1} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}
$$

and the corresponding simplex tableau is:

	x_1^+	x_1^-	x_2^+	x_2^-	x_3^+	$\mathbf{x_3^-}$	$\mathbf{x_4}$	$\mathbf{x_5}$	$\mathbf{x_6}$	$\mathbf{x_7}$
0	0	0	−1	1	0	0	0	0	0	0
8	3	−3	4	−4	−1	1	0	0	0	0
2	−1	1	1	−1	0	0	1	0	0	0
27	4	−4	6	−6	0	0	0	1	0	0
3	4	−4	−10	10	0	0	0	0	1	0
3	0	0	1	−1	0	0	0	0	0	1

Unsurprisingly, the current vertex is not yet optimal, we need to pivot x_2^+ into the basis, x_4 will leave the basis (we could also have chosen x_3^- for that). The new tableau is then:

	x_1^+	x_1^-	$\mathbf{x_2^+}$	x_2^-	x_3^+	$\mathbf{x_3^-}$	x_4	$\mathbf{x_5}$	$\mathbf{x_6}$	$\mathbf{x_7}$
2	−1	1	0	0	0	0	1	0	0	0
0	7	−7	0	0	−1	1	−4	0	0	0
2	−1	1	1	−1	0	0	1	0	0	0
15	10	−10	0	0	0	0	−6	1	0	0
23	−6	6	0	0	0	0	10	0	1	0
1	1	−1	0	0	0	0	−1	0	0	1

This is still not optimal, we can choose element $(1, 1)$ as the pivot element (thus, x_1^+ will enter the basis while x_3^- will leave) and get:

	$\mathbf{x_1^+}$	x_1^-	$\mathbf{x_2^+}$	x_2^-	x_3^+	x_3^-	x_4	$\mathbf{x_5}$	$\mathbf{x_6}$	$\mathbf{x_7}$
2	0	0	0	0	$-1/7$	$1/7$	$3/7$	0	0	0
0	1	−1	0	0	$-1/7$	$1/7$	$-4/7$	0	0	0
2	0	0	1	−1	$-1/7$	$1/7$	$3/7$	0	0	0
15	0	0	0	0	$10/7$	$-10/7$	$-2/7$	1	0	0
23	0	0	0	0	$-6/7$	$6/7$	$46/7$	0	1	0
1	0	0	0	0	$1/7$	$-1/7$	$-3/7$	0	0	1

Now, x_3^+ will enter the basis while x_7 will leave:

	$\mathbf{x_1^+}$	x_1^-	$\mathbf{x_2^+}$	x_2^-	$\mathbf{x_3^+}$	x_3^-	x_4	$\mathbf{x_5}$	$\mathbf{x_6}$	x_7
3	0	0	0	0	0	0	0	0	0	1
1	1	−1	0	0	0	0	−1	0	0	1
3	0	0	1	−1	0	0	0	0	0	1
5	0	0	0	0	0	0	4	1	0	−10
29	0	0	0	0	0	0	4	0	1	6
7	0	0	0	0	1	−1	−3	0	0	7

The solution is now LP optimal: $x_1^+ = 1$, $x_2 = 3$, $x_3^+ = 7$, $x_5 = 5$, $x_6 = 29$, all other variables are 0. As this is also an integer solution, this completes the Gomory cutting planes algorithm and we have obtained our final solution. In the original problem this corresponds to the solution $x^* = (1, 3)^T$.

Problem 5.3 (Branch & Bound for Traveling Salesman)

Consider the optimization version of the traveling salesman problem (TSP). We want to solve this problem using a branch & bound approach that does not originate from an ILP formulation of the problem. Let $G = K_n$ be the complete graph on the node set $V = \{1, \ldots, n\}$ and $c : E \to \mathbb{N}$ a distance function on the edges of G.

a) A 1-*tree* on G is a subgraph consisting of a spanning tree of the nodes $\{2, \ldots, n\}$ and two edges incident with node 1. Show that the value

$$L(n, c) := \min \left\{ c(T) = \sum_{e \in T} c_e : T \text{ is a 1-tree} \right\}$$

is a lower bound for the length of a minimum traveling salesman tour on G.

b) How can you compute $L(n, c)$ in polynomial time?

c) Use your 1-tree algorithm as a bounding scheme and the following branching scheme to design a branch & bound algorithm for TSP. Outline the whole algorithm and give details on how to compute solutions or relaxations of the subproblems.

Branching Scheme A subproblem of the TSP on G will be denoted by two disjoint subsets $E_{\text{in}}, E_{\text{out}} \subset E$ of the edge set E. A subproblem $(E_{\text{in}}, E_{\text{out}})$ asks for a minimum distance tour on the node set V that uses all edges in E_{in} and none of the edges in E_{out}. For each subproblem, choose an edge $e \in E$ that has not been included in E_{in} or E_{out} of that subproblem and define a branching by considering the two subproblems including and excluding e, respectively.

d) Compare the following alternative branching scheme. How would you have to change your algorithm to incorporate it? Can you think of other possible branching schemes?

Branching Scheme A subproblem of the TSP on G will be denoted by two disjoint subsets $E_{\text{in}}, E_{\text{out}} \subset E$ of the edge set E. A subproblem $(E_{\text{in}}, E_{\text{out}})$ asks for a minimum distance tour on the node set V that uses all edges in E_{in} and none of the edges in E_{out}. For each subproblem, compute a 1-tree respecting these constraints. If the 1-tree is not a tour, there must be a node of degree at least 3. For each of the edges adjacent to this node, define a new subproblem by excluding that edge.

Answer to Problem 5.3

a) Let τ be a minimum TSP tour on G, then τ visits node 1, thus it contains two edges incident with that node. Removing these edges yields a spanning tree of the node subset $\{2, \ldots, n\}$, thus τ is a 1-tree. Of course, the length of a minimum 1-tree is a lower bound for the length of τ.

b) A minimum spanning tree on $\{2, \ldots, n\}$ can be computed by Prim's or Kruskal's algorithm in polynomial time. To obtain a minimum 1-tree, we simply add the two minimum weight edges incident to node 1, which can easily determined by an enumeration of the edges. This yields a polynomial time algorithm for computing a minimum 1-tree.

c) We will first present an approach that utilizes the LP relaxation of the TSP integer linear programming formulation. Start by solving the LP relaxation on the graph G. If the solution is integer, stop. Otherwise, choose an edge e where the associated edge variable is fractional and create two subproblems by adding the edge to E_{in} and to E_{out}, respectively. In the subproblems, the sets E_{in} and E_{out} can simply be respected by fixing the corresponding variables to 1 and to 0, respectively. For the subproblems, a lower bound is computed by the 1-tree heuristic. Additionally, feasible solutions are generated by a Christofides' algorithm for each node, so that good solutions should be found quickly and the pruning of nodes is enabled.

d) The second approach we present works without the help of linear programming altogether. We start by computing a minimum 1-tree on the graph G. If this already is a tour, we are done. Otherwise, there must be a node of degree at least 3 and we create a new subproblem for each edge adjacent to this node that is used in the 1-tree by adding that edge to E_{out} (the set E_{in} is not used in this approach).

   We proceed to solve the subproblems by again computing a minimum 1-tree. If its value is at least as high as that of the current best solution, then the node is pruned. Otherwise, if the 1-tree is a TSP tour, we have an optimal solution to that subproblem. We compare it to our current best solution and either uupdate that solution or simply remove the node from the set of active sets without storing any further information. If the minimum 1-tree is not a TSP tour and the node cannot be pruned, we proceed by branching in the same way as outlined above.

   We still need to give details on how to handle E_{out} during the solving process. To this end, we simply set the weight of an edge $e \in E_{\text{out}}$ to $c(e) := (n+1)c_{\max}$, so it will never be used in a minimum 1-tree if possible. If it still is used (which can be recognized from the objective value), the node is actually infeasible and is removed from the list of active nodes without branching.

Problem 5.4 (The Simplex Tableau Revisited)
Consider the following linear program:

$$\begin{aligned} \max\ & 5x_1 + 6x_2 \\ & 3x_1 + 5x_2 \leq 15 \\ & 3x_1 - 5x_2 \leq 0 \\ & x \geq 0 \end{aligned}$$

a) Sketch the feasible region for the above LP and guess an optimal solution.

b) Show how to compute the solution using the simplex method in tableau form.

Answer to Problem 5.4

a) The sketch is depicted in Figure 4

   The point $x^* = (2.5, 1.5)$ is feasible and seems to be the optimal solution for the LP.

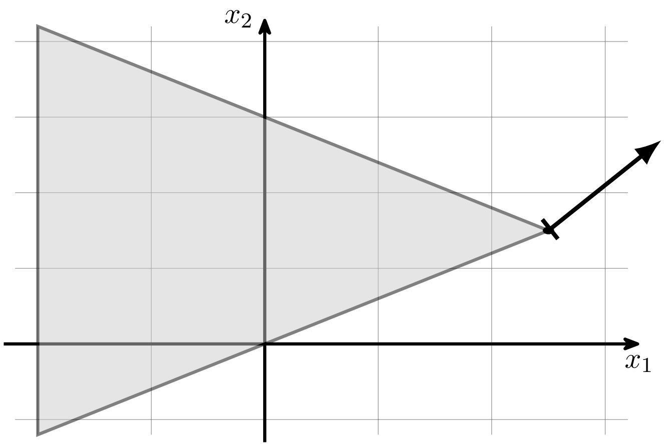

Figure 4: Feasible region of the LP.

b) We need to start by bringing the LP into canonical form (i. e. $Ax = b, x \geq 0$). The nonnegativity conditions are already there, so we will just need to introduce two additional slack variables x_3, x_4 toget equality constraints. The reformulated problem looks like this:

$$\begin{aligned} \max\ & 5x_1 + 6x_2 \\ & 3x_1 + 5x_2 + x_3 = 15 \\ & 3x_1 - 5x_2 + x_4 = 0 \\ & x \geq 0 \end{aligned}$$

This also has the advantage of directly giving us a starting solution: Simply choose $B = \{3, 4\}$ as a basis and $x_3 = 15$, $x_4 = 0$ as the basic feasible solution. The simplex tableau for the problem max $c^T x$ subject to $Ax = b$, $x \geq 0$ in general looks like this:

$c^T x$	$c_B^T A_B^{-1} A - c$
$A_B^{-1} b$	$A_B^{-1} A$

Starting with the vertex $x = (0, 0, 15, 0)^T$ and corresponding basis $B = \{3, 4\}$, we get

$$A_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and thus the simplex tableau looks like this:

	x_1	x_2	$\mathbf{x_3}$	$\mathbf{x_4}$
0	-5	-6	0	0
15	3	5	1	0
0	3	-5	0	1

For a pivot step, we start by checking for columns with negative reduced cost. In this case, columns one and two are candidates for entering the basis. We use the rule to always take the most negative entry, so we will bring column two (and thus x_2) into the basis. To see which element will have to leave the basis, we determine the row i that takes the value

$$\min_{i:(A_B^{-1}A)_{i,2}>0} \frac{(A_B^{-1}b)_i}{(A_B^{-1}A)_{i,2}}$$

where 2 is our pivot column. (Note that if there are no positive values here then the problem will be unbounded.) In the tableau, we just divide each positive entry in the pivot column by the entry in the leftmost column of the simplex tableau. The row with the minimal quotient will be our pivot element (and the corresponding basis element will have to leave the basis). In our tableau, we have only one positive entry, namely the one in row 1. As this row corresponds to our current first basis entry 3 the variable x_3 will have to leave the basis.

To actually perform the basis exchange, we use Gaussian elimination with the pivot element at position $(1, 2)$ to change column 2 into a unit vector with 1-entry at position $(1, 2)$ (basically, we do the matrix inversion in this step). Then, we use elementary row operations to obtain a reduced cost value of 0 in column 2 to compute the new reduced cost vector. This yields the following new simplex tableau:

	x_1	$\mathbf{x_2}$	x_3	$\mathbf{x_4}$
18	$-7/5$	0	$6/5$	0
3	$3/5$	1	$1/5$	0
15	6	0	1	1

The reduced cost row still has one negative entry, now in column one. Thus this will be our next pivot column, and to determine the pivot element we have to compute the minimizer of

$$\min_{i:(A_B^{-1}A)_{i,1}>0} \frac{(A_B^{-1}b)_i}{(A_B^{-1}A)_{i,1}}$$

There are now two positive entries in the column, thus we actually need to do the computations and obtain

$$\frac{3}{3/5} = 5 \quad \text{and} \quad \frac{15}{6} = \frac{5}{2},$$

thus the second row minimizes the expression. Our pivot element is therefore $(2, 1)$, meaning that x_1 will enter the basis while x_4 will leave it. We again do the computations using Gaussian elimination and get the following new tableau:

	$\mathbf{x_1}$	$\mathbf{x_2}$	x_3	x_4
$\frac{43}{2}$	0	0	$43/30$	$7/30$
$3/2$	0	1	$1/10$	$-1/10$
$5/2$	1	0	$1/6$	$1/6$

The reduced cost row is now nonnegative, meaning that no better objective value is possible anymore. Hence the current solution is optimal, and we can read off the solution $x_1 = 5/2$, $x_2 = 3/2$ and $x_3 = x_4 = 0$, which coincides nicely with our findings from the sketch. The value in the top left position of the simplex tableau is the optimal objective value, in this case $43/2$.

This last problem is provided to help you recap the simplex tableau. Solutions will shortly be available on the website.