Discrete Optimization (MA 3502)

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Exercise Sheet 5

Problem 5.1

Use the Gomory cutting plane algorithm to compute an optimal solution of the follwing ILP. Draw a sketch illustrating the situation in each step.

$$\begin{aligned} \max\ & 5x_1 + 3x_2 \\ & 2x_1 + 3x_2 \leq 10, \\ & x_1 - 2x_2 \leq 0 \\ & x \in \mathbb{Z}^2 \end{aligned}$$

Problem 5.2

Consider the polyhedron $P = \{x \in \mathbb{R}^2 : Ax \leq b\}$ given by

$$A := \begin{pmatrix} -3 & -4 \\ -1 & 1 \\ 4 & 6 \\ 4 & -10 \end{pmatrix}, \quad b := \begin{pmatrix} -8 \\ 2 \\ 27 \\ 3 \end{pmatrix}.$$

A $\mathcal{V}$-presentation of P is given by

$$P = \operatorname{conv}\left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} \frac{3}{2} \\ \frac{7}{2} \end{pmatrix}, \begin{pmatrix} \frac{9}{2} \\ \frac{3}{2} \end{pmatrix}, \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix} \right\}.$$

a) Draw a sketch of both P and its integer hull P_I.

b) Can you determine some proper cut for P_I just by "staring" at the inequalities?

c) Consider the vertex $x^* = (3/2, 7/2)^T$ of P. Compute a Gomory cut at x^* with respect to the first component.

d) Using $B = \{2, 3\}$ as a starting basis corresponding to x^*, use the Gomory cutting plane algorithm to compute the optimal solution to the following ILP:

$$\begin{aligned} \max\ & x_2 \\ & Ax \leq b \\ & x \in \mathbb{Z}^2 \end{aligned}$$

Please turn over.

Problem 5.3 (Branch & Bound for TRAVELING SALESMAN)
Consider the optimization version of the traveling salesman problem (TSP). We want to solve this problem using a branch & bound approach that does not originate from an ILP formulation of the problem. Let $G = K_n$ be the complete graph on the node set $V = \{1, \dots, n\}$ and $c : E \to \mathbb{N}$ a distance function on the edges of G.

a) A 1-*tree* on G is a subgraph consisting of a spanning tree of the nodes $\{2, \dots, n\}$ and two edges incident with node 1. Show that the value

$$L(n, c) := \min \left\{ c(T) = \sum_{e \in T} c_e : T \text{ is a 1-tree} \right\}$$

is a lower bound for the length of a minimum traveling salesman tour on G.

b) How can you compute $L(n, c)$ in polynomial time?

c) Use your 1-tree algorithm as a bounding scheme and the following branching scheme to design a branch & bound algorithm for TSP. Outline the whole algorithm and give details on how to compute solutions or relaxations of the subproblems.

Branching Scheme A subproblem of the TSP on G will be denoted by two disjoint subsets $E_{\text{in}}, E_{\text{out}} \subset E$ of the edge set E. A subproblem $(E_{\text{in}}, E_{\text{out}})$ asks for a minimum distance tour on the node set V that uses all edges in E_{in} and none of the edges in E_{out}. For each subproblem, choose an edge $e \in E$ that has not been included in E_{in} or E_{out} of that subproblem and define a branching by considering the two subproblems including and excluding e, respectively.

d) Compare the following alternative branching scheme. How would you have to change your algorithm to incorporate it? Can you think of other possible branching schemes?

Branching Scheme A subproblem of the TSP on G will be denoted by two disjoint subsets $E_{\text{in}}, E_{\text{out}} \subset E$ of the edge set E. A subproblem $(E_{\text{in}}, E_{\text{out}})$ asks for a minimum distance tour on the node set V that uses all edges in E_{in} and none of the edges in E_{out}. For each subproblem, compute a 1-tree respecting these constraints. If the 1-tree is not a tour, there must be a node of degree at least 3. For each of the edges adjacent to this node, define a new subproblem by excluding that edge.

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Problem 5.4 (The Simplex Tableau Revisited)
Consider the following linear program:

$$\begin{aligned} \max\ & 5x_1 + 6x_2 \\ & 3x_1 + 5x_2 \le 15 \\ & 3x_1 - 5x_2 \le 0 \\ & x \ge 0 \end{aligned}$$

a) Sketch the feasible region for the above LP and guess an optimal solution.

b) Show how to compute the solution using the simplex method in tableau form.

This last problem is provided to help you recap the simplex tableau. Solutions will shortly be available on the website.