Discrete Optimization (MA 3502)

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Exercise Sheet 6

Problem 6.1 (Cutting Stock)
Consider the cutting stock problem discussed in the lecture: Several orders of small paper rolls have to be produced by cutting larger rolls. The task is to devise a cutting strategy that uses as few large rolls as possible. Consider the following example where the large rolls have a width of $W = 94$ and the paper mill has the following orders:

width w_i	demand d_i
17	150
21	96
22.5	48
24	108
29.5	225

a) Write down the cutting stock LP and its dual. What does the pricing problem look like (i. e., what is the reduced cost)? If necessary, have a look at the scribbles for December 22nd that have been extended by one page that shows the pricing problem.

b) Start with the five cutting patterns that cut a large paper roll into smaller rolls of equal width and solve the master LP restricted to these variables.

c) Compute the corresponding dual values and write down the column generation pricing ILP.

d) Solve the column generation ILP. What new cutting pattern does this produce?

Answer to Problem 6.1

a) Let us first recall the "cutting pattern formulation" of the cutting stock problem: Let Q denote the set of cutting patterns and let a cutting pattern x^j be a vector in $\mathbb{N}_0^5$ where x_i^j indicates how many rolls of width w_i will be cut out of a large roll for pattern x^j. With variables $u^j \in \mathbb{N}_0$ indicating how many times cutting pattern x^j is applied, we get the following problem formulation:

$$\begin{aligned} \min \sum_{j=1}^{|Q|} u^j & \\ \sum_{j=1}^{|Q|} x_i^j u^j \geq d_i & \quad \forall i = 1, \ldots, 5 \\ u^j \in \mathbb{N}_0 & \quad \forall j \in Q \end{aligned}$$

The dual of the LP relaxation of this problem with dual variables $y_i, i = 1, \ldots, 5$, is

$$\begin{aligned} \max \ & \sum_{i=1}^{5} d_i y_i \\ & \sum_{i=1}^{5} x_i^j y_i \le 1 \quad \forall j \in Q \\ & y_i \ge 0 \quad \forall i = 1, \ldots, 5. \end{aligned}$$

In the pricing problem we are looking for a violated dual constraint, i. e. for a given dual solution y we need to find a cutting pattern x^j such that

$$\bar{c}^j := 1 - \sum_{i=1}^{5} x_i^j y_i < 0,$$

(this expression is known as "reduced cost") where the y variables correspond to u through complementary slackness (or simply through $y_B = A_B^{-1} c_B$ for the basic variables, $y_N = 0$ for the non-basic ones). The pricing problem is therefore

$$\begin{aligned} \min \ & (1 - \sum_{i=1}^{5} x_i^j y_i) \\ & \sum_{i=1}^{5} x_i^j w_i \le W \\ & x_i^j \in \mathbb{N}_0 \quad \forall i = 1, \ldots, 5. \end{aligned}$$

A new column and a corresponding new cutting pattern has been found if the pricing yields an objective value of strictly less than 0. Otherwise, the current solution is optimal and the column generation process terminates.

b) We start with the cutting patterns

$$\begin{aligned} x^1 &= (5, 0, 0, 0, 0)^T, & x^4 &= (0, 0, 0, 3, 0)^T, \\ x^2 &= (0, 4, 0, 0, 0)^T, & x^5 &= (0, 0, 0, 0, 3)^T, \\ x^3 &= (0, 0, 4, 0, 0)^T. \end{aligned}$$

The restricted master LP is then

$$\begin{aligned} \min \ & u_1 + \ldots + u_5 \\ & 5u_1 \ge 150 \\ & 4u_2 \ge 96 \\ & 4u_3 \ge 48 \\ & 3u_4 \ge 108 \\ & 3u_5 \ge 225 \\ & u \ge 0. \end{aligned}$$

The optimal solution can of course be immediately seen: $u = (30, 24, 12, 36, 75)^T$.

c) As all $u^j > 0$ are strictly positive, we can use the complementary slackness conditions to compute the values for the corresponding dual variables y_i, getting

$$y = \left(\frac{1}{5}, \frac{1}{4}, \frac{1}{4}, \frac{1}{3}, \frac{1}{3}\right)^T.$$

This leads to the following pricing problem:

$$\min\ (1 - \sum_{i=1}^{5} x_i y_i + 1 = 1 - \frac{x_1}{5} + \frac{x_2}{4} + \frac{x_3}{4} + \frac{x_4}{3} + \frac{x_5}{3})$$
$$\sum_{i=1}^{n} w_i x_i = 17x_1 + 21x_2 + 22.5x_3 + 24x_4 + 29.5x_5 \leq 94$$
$$x \in \mathbb{N}_0^5$$

Notice that now x is the vector of decision variables and y is considered constant.

d) An optimal solution to the above knapsack problem (the minimization is really a maximization due to the negative sign) is

$$x^* = (0, 1, 0, 3, 0)^T$$

with objective value $-\frac{1}{4}$. The reduced cost of this new cutting pattern (the objective value) is negative, therefore incorporating this cutting pattern into the set of admissible patterns (and adding a corresponding u^j-variable) could lead to better results.

The procedure may be repeated, producing more cutting patterns, until an optimal LP solution is reached.

Problem 6.2 (Column Generation for Multicommodity Flow)
Let $G = (V, A)$ be a directed graph on n nodes and $b : A \to \mathbb{N}$ a capacity function on the arcs of G. Furthermore, let $K \in \mathbb{N}$ be a number of "commodities" and for each commodity k let $s_k, t_k \in V$ be a source and target node, $d_k \in \mathbb{N}$ a demand and $c^k : A \to \mathbb{N}$ a cost function on the arcs of G. The *multicommodity flow problem (MCF)* asks for K flows $f^1, \ldots, f^K$ in the graph G such that

- each flow f^k is an $s_k - t_k$-flow of value d_k,
- the combined flow for each arc $(i, j) \in A$ is bounded by b_{ij} and
- the total cost over all flows is minimized.

a) Devise an LP-model for MCF using variables f_{ij}^k denoting the flow of commodity k along arc (i, j).

b) Devise an LP-model for MCF using variables P_ℓ^k denoting the flow of commodity k along the ℓ-th element of $\mathcal{P}^k$, where $\mathcal{P}^k$ is a collection of all possible $s_k - t_k$ paths in G. What are the main differences of the two models? Which one would you use in practice and why?

c) Using duality theory and the complementary slackness conditions, characterize the optimal solutions of the path-based formulation.

d) Devise a column generation procedure for the path-based formulation. Give precise instructions on how to implement the pricing step! (Can you come up with a combinatorial algorithm for the pricing step?)

Answer to Problem 6.2

a) The flow ILP formulation is quite "classical". It simply combines k standard flow formulations, tied together via a single constraint for every edge (and, of course, via summing up the total cost in the objective function):

$$\begin{aligned}
\min \sum_k \sum_{e \in E} c_e^k \cdot f_e^k & \\
\sum_k f_e^k \le b_e & \quad \text{for all } e \in E \\
\sum_{j:(i,j)\in E} f_{ij}^k - \sum_{j:(j,i)\in E} f_{ji}^k = 0 & \quad \text{for all } i \in V \setminus \{s_k, t_k\} \\
\sum_{j:(s,j)\in E} f_{sj}^k = d_k & \\
f \ge 0 &
\end{aligned}$$

Note that we do not need a flow constraint for the node t_k, this is implied by the constraint for s_k and by the flow conservation constraint on all other nodes. Let us also remark that the multicommodity flow problem is $\mathcal{NP}$-hard in general if we ask for integer flows.

b) For clarity of exposition, let us deviate from the notation given above a little: We will denote the set of all s_k-t_k-paths by $\mathcal{P}^k$, the flow on any path P will be denoted by $f(P)$, the cost of all the edges along path P will be written als $c(P)$. We then get the following *path flow formulation* for multicommodity flow:

$$\begin{aligned}
\min \sum_k \sum_{P \in \mathcal{P}^k} c(P) \cdot f(P) & \\
\sum_k \sum_{P \in \mathcal{P}^k : e \in P} -f(P) \ge -b_e & \quad \text{for all } e \in E && (1) \\
\sum_{P \in \mathcal{P}^k} f(P) \ge d_k & \quad \text{for all } k && (2) \\
f \ge 0 &
\end{aligned}$$

Note that we demand at least a flow of d_k for each commodity instead of exactly a flow of d_k in (2). As we minimize with positive cost terms, this will not change the optimal solution, but it will help us formulate the dual LP more conveniently. This is also the reason for "flipping" the edge inqualities (1) by multiplying them by (-1).

As for the main differences, the flow model does have considerably more constraints than the path-bases formulation. However, the latter has an exponential number of variables, which would usually be a "no go" in integer programming, at least for networks of more than trivially solvable sizes. On the other hand, only a very small number of these variables will actually carry a positive flow in any basic optimal solution (and these are the solutions the simplex algorithm will give us), making the path-based formulation an ideal candidate for column generation. We only have to figure out an efficient way of generating new columns, which we will turn to in the next part of this exercise.

c) We start by formulating the dual problem. Let us assign dual variables w_e to the constraints

(1) and dual variables σ_k to the constraints (2). The LP dual then becomes

$$\begin{aligned} \max \sum_k d_k \cdot \sigma_k - \sum_{e \in E} b_e \cdot w_e & \\ \sigma_k - \sum_{e \in P} w_e \leq c^k(P) & \quad \text{for all } k \text{ and all } P \in \mathcal{P}^k \\ w, \sigma \geq 0. & \end{aligned}$$

For checking the optimality of our solution, the complementary slackness conditions yield a handy criterion: A primal-dual feasible pair f, w, σ is optimal if and only if the following hold:

$$\begin{aligned} \left[c^k(P) - \sigma_k + \sum_{e \in P} w_e\right] \cdot f(P) = 0 & \qquad \text{for all } k \text{ and all } P \in \mathcal{P}^k \\ \left[b_e - \sum_k \sum_{P \in \mathcal{P}^k : e \in P} f(P)\right] \cdot w_e = 0 & \qquad \text{for all } e \in E \\ \left[\sum_{P \in \mathcal{P}^k} f(P) - d_k\right] \cdot \sigma_k = 0 & \qquad \text{for all } k \end{aligned}$$

d) Searching for a violated dual inequality amounts to find a path $P\mathcal{P}^k$ where the reduced cost satisfies

$$\overline{c}(P) := c^k(P) + \sum_{e \in P} w_e - \sigma_k < 0,$$

and a solution is optimal once $\overline{c}(P) \geq 0$ for all paths $P \in \mathcal{P}^k$ and all k. For a column generation scheme we have to be able to determine some k and some $P \in \mathcal{P}^k$ with negative reduced cost without going through all possible paths. Observe that for some fixed good k the reduced cost can be written as

$$\overline{c}(P) = c^k(P) + \sum_{e \in P} w_e - \sigma_k = \sum_{e \in P} (c_e^k + w_e) - \sigma_k,$$

where σ_k is constant for all paths $P \in \mathcal{P}^k$. Note that the cost of the path can be written as a sum of the costs over the individual edges. Thus in order to determine a path with negative reduced cost, we can just determine the shortest path for some fixed commodity k, where the costs of edge edge e are set to $c_e^k + w_e$ for the current dual solution (w, σ). We then substract σ_k from the cost of this shortest path. If the value is negative, we have found a variable that should be added to the problem, if not, no path for good k can improve the solution. We try this procedure for every commodity, until we either find a suitable path (and corresponding primal variable) or have a proof of optimality. This pricing algorithm requires the solution of at most k shortest path problems, which can be computed in polynomial time (e. g., using Dijkstra's algorithm) and some fiddling with the edge costs, thus yielding an efficient pricing step for our column generation algorithm.

Problem 6.3 (Extended Knapsack Cover Inequalities)
Let $a, p \in \mathbb{R}^n$, $b \in \mathbb{N}$ be a knapsack problem with size vector a and profit vector p and let

$$\mathcal{K}(a, b) := \operatorname{conv} \left\{ x \in \{0,1\}^n : a^T x \leq b \right\}$$

denote the *knapsack polytope*. In the lecture, you introduced the *knapsack cover inequalities*

$$\sum_{i\in C} x_i \leq |C| - 1 \tag{KNAP-C}$$

for a *cover* $C \subset [n]$, i. e. $\sum_{i\in C} a_i > b$.

a) Give an example that shows that the inequalities (KNAP-C) do not generally define a facet of $\mathcal{K}(a, b)$.

b) For a cover $C \subset [n]$, define the *extended cover*

$$E(C) := C \cup \{j \in \{1, \dots, n\} : a_j \geq a_i \ \forall i \in C\}.$$

Derive a valid inequality for $\mathcal{K}(a, b)$ that involves $E(C)$. Give an example where your extended cover inequality is stronger than the cover inequality.

c) Do the extended cover inequalities always define a facet of $\mathcal{K}(a, b)$? Give either a short proof or a counterexample

d) Can you find instances of the knapsack problem with the following properties?

   i) The LP relaxation yields a non-integral optimal solution, but addition of some cover inequality leads to an integral optimal solution.

   ii) The LP relaxation yields a non-integral optimal solution, adding a cover inequality still does not suffice for integrality, but adding the corresponding extended cover inequality does.

e) *Bonus exercise:* Use a programming language of your choice to implement a program that does the following:

   i) compute a solution for the LP relaxation of an instance of the knapsack problem

   ii) as long as the the solution is not integral, separate some cover inequality (or assert there is no violated cover inequality), add it (or its extension) to the relaxation and resolve. You will need to figure out a procedure to separate a violated cover inequality first.

   As you will need an LP solver, Xpress Mosel or MatLab is recommended for this exercise.

Answer to Problem 6.3

a) Consider the three-dimensional knapsack problem $x_1 + x_2 + x_3 \leq 1$. Then $C = \{1, 2\}$ is a cover, but the corresponding inequality

$$x_1 + x_2 \leq 1$$

does not define a facet of $\mathcal{K}((1, 1, 1)^T, 1)$, because it is strictly dominated by the valid inequality $x_1 + x_2 + x_3 \leq 1$.

b) Let C be a cover and $E(C)$ the corresponding extended cover, then the inequality

$$\sum_{j\in E(C)} x_j \leq |C| - 1$$

is valid for the knapsack polytope. To see this, assume there was a vertex $x \in \{0, 1\}^n$ of $\mathcal{K}(a, b)$ that violated the above inequality. Then

$$\sum_{j\in E(C)} x_j \geq |C|$$

and thus at least $|C|$ of the x_j with $j \in E(C)$ are equal to 1. Therefore,

$$\sum_{j=1}^{n} a_j x_j \geq \sum_{j \in E(C)} a_j x_j = \sum_{j \in C} a_j x_j + \sum_{j \in E(C) \setminus C} x_j \geq \sum_{j \in C} a_j > b,$$

which shows that x is infeasible, in contradiction to $x \in \mathcal{K}(a, b)$.

Consider again the knapsack polytope $\mathcal{K}((1,1,1)^T, 1)$ and the cover $C = \{1, 2\}$, then $E(C) = \{1, 2, 3\}$ and the extended cover inequality $x_1 + x_2 + x_3 \leq 1$ is clearly stronger than the original cover inequality.

c) Consider the knapsack problem defined by $2x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 \leq 5$, then $C = \{4, 5\}$ is clearly a cover and $E(C) = \{4, 5\}$. The corresponding extended cover inequality is

$$x_4 + x_5 \leq 1.$$

As $x_4 + x_5 = 1$ implies $2x_1 + 2x_2 + 3x_3 \leq 1$, we get $x_1 = x_2 = x_3 = 0$ for every vertex in $\left\{x \in \mathcal{K} := \mathcal{K}((2,2,3,4,5)^T, 5) : x_4 + x_5 = 1\right\}$, hence this set is at most $5 - 3 = 2$-dimensional. As $\mathcal{K}$ is fully-dimensional (i. e., 5-dimensional), the inequality cannot induce a facet.

d) i) Consider the knapsack problem

$$\begin{aligned} \max\ & x_1 + 2x_2 \\ & 2x_1 + 2x_2 \leq 3 \\ & x \in \{0,1\}^2 \end{aligned}$$

For the LP-relaxation, the solution $x = (0.5, 1)^T$ is feasible and has an objective value of 2.5, clearly better than any integral solution. Consider the cover $C = \{1, 2\}$, then the cover inequality is $x_1 + x_2 \leq 1$, cutting off the above fractional solution. The optimum would now be $x = (0, 1)^T$, which is indeed the optimal integer solution of the knapsack problem.

ii) Consider

$$\begin{aligned} \max\ & x_1 + x_2 + 2x_2 \\ & 3x_1 + 3x_2 + 4x_3 \leq 5 \\ & x \in \{0,1\}^3, \end{aligned}$$

where a possible optimum of the LP-relaxation is $x = (1/3, 0, 1)^T$ with an objective value of $2 1/3$. Again, $C = \{2, 3\}$ is a possible cover, but the cover inequality $x_2 + x_3 \leq 1$ does not exclude the fractional solution. However, with the extended cover $E(C) = \{1, 2, 3\}$ we cut off this solution and would now get the integer optimum $x = (0, 0, 1)^T$ with an objective value of 2.

e) We will only discuss the separation of valid cover inequalities here, the implementation is left as an exercise for the reader. Given some (possibly fractional) solution x^* of the LP-relaxation of the knapsack problem $\mathcal{K}(a, b)$, we want to find a set $C \subset [n]$ where

$$\sum_{j \in C} a_j > B \quad \text{and} \quad \sum_{j \in C} x_j^* > |C| - 1.$$

Let us formulate that as a 0-1-optimization problem with a set of variables representing the cover we are looking for, i. e.,

$$z_j = \begin{cases} 1, & \text{if } j \in C \\ 0, & \text{otherwise.} \end{cases}$$

Then the search for a violated cover inequality can be expressed as the ILP

$$\max \sum_{j=1}^{n} (x_j^* - 1) z_j$$
$$\sum_{j=1}^{n} a_j z_j \geq B + 1.$$

Note that "$> B$" is equivalent to "$\geq B + 1$", because we can assume without loss of generality that all a_j are positive integers. Then an optimal solution z^* corresponds to a violated cover inequality if and only if the objective value is strictly greater than -1. Unfortunately, this problem is basically (after some variable transformations) a knapsack problem—so we have to solve a (number of) knapsack problem(s) in order to solve a knapsack problem, which admittedly sounds a little absurd. However, notice that the values of our items are small, so we can employ some dynamic programming scheme whose running time is dependent on $v_{\max}$ to get a practically fast algorithm for solving the problem. However, a more important aspect is that the idea may be employed for knapsack constraints that are part of some larger ILP comprising of many more constraints in addition to the knapsack. Using the idea of cover inequalities, one can obtain effective cutting planes for such problems.

Problem 6.4 (The Stable Set Polytope)
Consider a graph $G = (V, E)$ on $n = |V| \geq 3$ nodes. A *hole* in G is a subgraph $H = (V_H, E_H)$ that is isomorphic to a cycle. The *Stable Set Polytope* on G is defined as

$$P_{\text{STAB}}(G) := \text{conv}\left\{x \in \{0,1\}^n : x \text{ is the incidence vector of a stable set in } G\right\}.$$

Show that the following *odd hole inequality* is valid for $P_{\text{STAB}}(G)$ for each hole (V_H, E_H) in G:

$$\sum_{i \in V_H} x_i \leq \left\lfloor \frac{|V_H|}{2} \right\rfloor$$

Answer to Problem 6.4
Let $x \in P_{\text{STAB}}(G)$ be a vertex of the stable set polytope, then in particular $x \in \{0,1\}^n$. Consider a hole (V_H, E_H) in G and assume the odd hole inequality was not valid for that hole.

Let us first consider a hole of even cardinality, i. e. with $|V_H|$ even. As x is integer, that means

$$\sum_{i \in V_H} x_i \geq \frac{|V_H|}{2} + 1.$$

The hole contains precisely $|E_H| = |V_H|$ edges (it is isomorphic to a cycle), thus there must be at least one edge $\{u, v\} \in E_H$ with $x_u = x_v = 1$, contradicting the stable set property.

Similarly, if $|V_H|$ is odd, we get

$$\sum_{i \in V_H} x_i \geq \frac{|V_H| - 1}{2} + 1$$

and the same contradiction follows. Thus the odd hole inequality holds for each hole. (It is called *odd* hole inequality, because it only provides a proper cut for odd holes.)