Discrete Optimization (MA 3502)

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Exercise Sheet 6

Problem 6.1 (Cutting Stock)
Consider the cutting stock problem discussed in the lecture: Several orders of small paper rolls have to be produced by cutting larger rolls. The task is to devise a cutting strategy that uses as few large rolls as possible. Consider the following example where the large rolls have a width of $W = 94$ and the paper mill has the following orders:

width w_i	**demand** d_i
17	150
21	96
22.5	48
24	108
29.5	225

a) Write down the cutting stock LP and its dual. What does the pricing problem look like (i. e., what is the reduced cost)? If necessary, have a look at the scribbles for December 22nd that have been extended by one page that shows the pricing problem.

b) Start with the five cutting patterns that cut a large paper roll into smaller rolls of equal width and solve the master LP restricted to these variables.

c) Compute the corresponding dual values and write down the column generation pricing ILP.

d) Solve the column generation ILP. What new cutting pattern does this produce?

Problem 6.2 (Column Generation for Multicommodity Flow)
Let $G = (V, A)$ be a directed graph on n nodes and $b : A \to \mathbb{N}$ a capacity function on the arcs of G. Furthermore, let $K \in \mathbb{N}$ be a number of "commodities" and for each commodity k let $s_k, t_k \in V$ be a source and target node, $d_k \in \mathbb{N}$ a demand and $c^k : A \to \mathbb{N}$ a cost function on the arcs of G. The *multicommodity flow problem (MCF)* asks for K flows $f^1, \dots, f^K$ in the graph G such that

- each flow f^k is an $s_k - t_k$-flow of value d_k,
- the combined flow for each arc $(i, j) \in A$ is bounded by b_{ij} and
- the total cost over all flows is minimized.

a) Devise an LP-model for MCF using variables f_{ij}^k denoting the flow of commodity k along arc (i, j).

b) Devise an LP-model for MCF using variables P_ℓ^k denoting the flow of commodity k along the ℓ-th element of $\mathcal{P}^k$, where $\mathcal{P}^k$ is a collection of all possible $s_k - t_k$ paths in G. What are the main differences of the two models? Which one would you use in practice and why?

c) Using duality theory and the complementary slackness conditions, characterize the optimal solutions of the path-based formulation.

Please turn over.

d) Devise a column generation procedure for the path-based formulation. Give precise instructions on how to implement the pricing step! (Can you come up with a combinatorial algorithm for the pricing step?)

Problem 6.3 (Extended Knapsack Cover Inequalities)
Let $a, p \in \mathbb{R}^n$, $b \in \mathbb{N}$ be a knapsack problem with size vector a and profit vector p and let

$$\mathcal{K}(a, b) := \text{conv}\left\{x \in \{0,1\}^n : a^T x \leq b\right\}$$

denote the *knapsack polytope*. In the lecture, you introduced the *knapsack cover inequalities*

$$\sum_{i \in C} x_i \leq |C| - 1 \qquad \text{(KNAP-C)}$$

for a *cover* $C \subset [n]$, i. e. $\sum_{i \in C} a_i > b$.

a) Give an example that shows that the inequalities (KNAP-C) do not generally define a facet of $\mathcal{K}(a, b)$.

b) For a cover $C \subset [n]$, define the *extended cover*

$$E(C) := C \cup \{j \in \{1, \ldots, n\} : a_j \geq a_i \; \forall i \in C\}.$$

Derive a valid inequality for $\mathcal{K}(a, b)$ that involves $E(C)$. Give an example where your extended cover inequality is stronger than the cover inequality.

c) Do the extended cover inequalities always define a facet of $\mathcal{K}(a, b)$? Give either a short proof or a counterexample

d) Can you find instances of the knapsack problem with the following properties?

   i) The LP relaxation yields a non-integral optimal solution, but addition of some cover inequality leads to an integral optimal solution.

   ii) The LP relaxation yields a non-integral optimal solution, adding a cover inequality still does not suffice for integrality, but adding the corresponding extended cover inequality does.

e) *Bonus exercise:* Use a programming language of your choice to implement a program that does the following:

   i) compute a solution for the LP relaxation of an instance of the knapsack problem

   ii) as long as the the solution is not integral, separate some cover inequality (or assert there is no violated cover inequality), add it (or its extension) to the relaxation and resolve. You will need to figure out a procedure to separate a violated cover inequality first.

   As you will need an LP solver, Xpress Mosel or MatLab is recommended for this exercise.

Problem 6.4 (The Stable Set Polytope)
Consider a graph $G = (V, E)$ on $n = |V| \geq 3$ nodes. A *hole* in G is a subgraph $H = (V_H, E_H)$ that is isomorphic to a cycle. The *Stable Set Polytope* on G is defined as

$$P_{\text{STAB}}(G) := \{x \in \{0,1\}^n : x \text{ is the incidence vector of a stable set in } G\}.$$

Show that the following *odd hole inequality* is valid for $P_{\text{STAB}}(G)$ for each hole (V_H, E_H) in G:

$$\sum_{i \in V_H} x_i \leq \left\lfloor \frac{|V_H|}{2} \right\rfloor$$