Due Monday, 26 Oct. 2015.

Recall the following notation:

- A set $S \subseteq \mathbb{R}^n$ is *path connected* iff, for any points $x, y \in S$, there is a continuous function $\phi : [0,1] \rightarrow S$ with $\phi(0)=x$ and $\phi(1)=y$.

Problems

0: Please read the posted pdfs on: (i) Hermite Factorization and (ii) the Principle of the Argument. **Note:** You need **not** do Problem 0 first: it may be more helpful do it in parallel with the others...

1: Let $H_{u,v}$ denote the number of complex roots of $e^{\sqrt{2}y}=1$ in the horizontal strip $\mathbb{R}\times[u,v]$. Please find a positive integer C with $\left|H_{u,v} - \sqrt{2}\cdot\frac{v-u}{2\pi}\right| \leq C$ for all $u<v$. **Hint:** First find an explicit formula for all the roots of $e^{\sqrt{2}y}=1$.

2: Use induction, and Rolle's Theorem, to prove the following upper bound:

> Any univariate polynomial f of the form $c_1x^{a_1} + \cdots + c_tx^{a_t}$, with $c_i \in \mathbb{R}^*$ for all i and $a_1 < \cdots < a_t$, has at most $t-1$ positive roots.

(René Descartes wrote a more refined statement in 1637 and we'll soon prove a more powerful extension.) Can you find, for any t, a univariate t-nomial with exactly $t-1$ (distinct) positive roots?

3: Suppose $f(x)=3^{\beta+\gamma}a - 3^{\beta}bx^{\alpha} + cx^{\alpha+\beta}$ for some $a, b, \alpha, \beta, \gamma \in \mathbb{N}$ with $\gamma > \alpha$ and $3 \not| a, b, c$. Please prove that if f has an integer root of the form $3^r s$ for some $r \in \mathbb{N}$ and $s \in \mathbb{Z}$, with s not divisible by 3, then there are just two possibilities for r. Please find these two possible values for r.

4: Please prove that, given any *finite* subset $X \subset \mathbb{R}^2$, the set $\mathbb{R}^2 \setminus X$ is path-connected. In particular, please outline your own method to build an explicit path in $\mathbb{R}^2 \setminus X$ connecting any two given points p and q. You should think of your construction as an algorithm you could actually implement. **Note:** I'm really just looking for a clear, concise explanation of a method of your own that works. Please keep it under 2 pages. (I've seen excellent solutions in 1 page.)

5: **(a)** Please prove that $\log\left|x^A\right| = (\log|x|)A$ for any $x=(x_1,\ldots,x_n) \in (\mathbb{C}^*)^n$.

(b) Please prove that $f(x):=x^A$ is invertible (as a function from $\mathbb{R}^n_+$ to $\mathbb{R}^n_+$) $\Longleftrightarrow$ $\det A \neq 0$. **Hint:** When $\det A \neq 0$ there is a very obvious choice for the inverse of f. The harder direction is then f invertible $\Longrightarrow \det A \neq 0$. A good approach is to prove the contrapositive: assume $\det A = 0$ and then prove that f must fail to be injective (which implies a failure of invertibility). In particular, when $\det A = 0$, you can use a nonzero vector in the left-null space of A to generate numerous *distinct* x with $x^A = (1,\ldots,1)$.

We usually call a substitution of the form $x := y^A$ a **monomial change of variables**.

NOTE: Please feel free to e-mail comments, questions, and/or corrections.