Due Monday, 26 Oct. 2015.

1: Let $H_{u,v}$ denote the number of complex roots of $e^{\sqrt{2}y}=1$ in the horizontal strip $\mathbb{R}\times[u,v]$. Please find a positive integer C with $\left|H_{u,v}-\sqrt{2}\cdot\frac{v-u}{2\pi}\right|\leq C$ for all $u<v$.

Solution

First we find the solutions of $e^{\sqrt{2}y}=1$, they are $\{\sqrt{2}k\pi i \mid k\in\mathbb{Z}\}$, all placed on the line $Re(y)=0$. So in every horizontal strip $\mathbb{R}\times[u,v]$ we have at most $\frac{v-u}{\sqrt{2}\pi}+1$ solutions. Hence, we can choose $C=1$.

2: Use induction, and Rolle's Theorem, to prove the following upper bound:

> Any univariate polynomial f of the form $c_1x^{a_1}+\cdots+c_tx^{a_t}$, with $c_i\in\mathbb{R}^*$ for all i and $a_1<\cdots<a_t$, has at most $t-1$ positive roots.

Can you find, for any t, a univariate t-nomial with exactly $t-1$ (distinct) positive roots?

Solution

We prove the statement by induction on t.

For $t=1$, $f(x)=c_1x^{a_1}$, so its only root is 0. For $t\geq 2$, we can assume $a_1=0$, as otherwise we could factor $f(x)=x^{a_1}(c_1+c_2x^{a_2-a_1},\ldots,c_tx^{a_t-a_1})$ and the factor x^{a_1} could be left out, as it does not contribute to the number of positive roots of f. Assume f has N distinct positive roots. They define $N-1$ open intervals where we can apply Rolle's theorem, and obtain that f' has at least $N-1$ positive roots. As f' has $t-1$ terms, by induction it must be $N-1\leq t-2$, which means $N\leq t-1$.

As t-nomials with exactly $t-1$ positive roots, choose $f(x):=(x-1)(x-2)\cdots(x-t+1)$.

3: Suppose $f(x)=3^{\beta+\gamma}a-3^{\beta}bx^{\alpha}+cx^{\alpha+\beta}$ for some $a,b,c,\alpha,\beta,\gamma\in\mathbb{N}$ with $\gamma>\alpha$ and $3\not|a,b,c$. Please prove that if f has an integer root of the form $3^r s$ for some $r\in\mathbb{N}$ and $s\in\mathbb{Z}$, with s not divisible by 3, then there are just two possibilities for r. Please find these two possible values for r.

Solution

If $f(3^r s)=3^{\beta+\gamma}a-3^{\beta}b(3^r s)^{\alpha}+c(3^r s)^{\alpha+\beta}=0$, then $3^{\beta+\gamma}a+3^{\alpha r+\beta r}cs^{\alpha+\beta}=3^{\alpha r+\beta}bs^{\alpha}$. Both sides must contain the same power of 3 in their factorization, and as $3\not|a,b,c,s$ it follows

$$min\{\beta+\gamma,\alpha r+\beta r\}=\alpha r+\beta$$

If $min\{\beta+\gamma,\alpha r+\beta r\}=\beta+\gamma$, then $r=\frac{\gamma}{\alpha}$, if $min\{\beta+\gamma,\alpha r+\beta r\}=\alpha r+\beta r$, then $r=1$.

4: Please prove that, given any *finite* subset $X\subset\mathbb{R}^2$, the set $\mathbb{R}^2\setminus X$ is path-connected. In particular, please outline your own method to build an explicit path in $\mathbb{R}^2\setminus X$ connecting any two given points p and q. You should think of your construction as an algorithm you could actually implement.

Solution

Suppose $p=(a,b),q=(c,d)\in\mathbb{R}^2\setminus X$, define a family of parabolas $\mathcal{P}_\alpha$ passing through p,q, that if $a\neq c$ are defined by the equations

$$y=\alpha x^2+\beta x+\gamma \qquad \alpha\in\mathbb{R}$$

where

$$\beta = \frac{d-b}{c-a} - \alpha(c+a) \qquad \gamma = b - \alpha a^2 - a\frac{d-b-\alpha(c^2-a^2)}{c-a}$$

(if $a = c$ use the family $x = \alpha y^2 + \beta y + \gamma$, and suitable β, γ instead). One can choose α so that $\mathcal{P}_\alpha$ does not pass through any point of X, and use the arc of parabola (p, q) as path.

5: **(a)** Please prove that $\log\left|x^A\right| = (\log|x|)A$ for any $x = (x_1, \ldots, x_n) \in (\mathbb{C}^*)^n$.
(b) Please prove that $f(x) := x^A$ is invertible (as a function from $\mathbb{R}^n_+$ to $\mathbb{R}^n_+$) $\Longleftrightarrow$ $\det A \neq 0$.

Solution

(a) Following the definition given in the lecture, denoting $c_1 \ldots, c_n$ the columns of $A \in \mathbb{R}^{n\times n}$, we have

$$\log|x^A| = \log|(x^{c_1}, \ldots, x^{c_n})| = (\log|x^{c_1}|, \ldots, \log|x^{c_n}|) =$$

$$= (\log|x_1^{a_{1,1}}| + \cdots + \log|x_n^{a_{n,1}}|, \ldots, \log|x_1^{a_{1,n}}| + \cdots + \log|x_n^{a_{n,n}}|) =$$

$$= (a_{1,1}\log|x_1| + \cdots + a_{n,1}\log|x_n|, \ldots, a_{1,n}\log|x_1| + \cdots + a_{n,n}\log|x_n|) = (\log|x|)A$$

(b) "$\Leftarrow$" If $det(A) \neq 0$, then define $f^{-1}(x) = x^{A^{-1}}$, that is the inverse of f by the proposition stated in class.
"$\Rightarrow$" We show that if $det(A) = 0$, then f is not invertible. Since A is not invertible, it has a nonzero left null vector $v = (v_1, ..., v_n)$. In particular, $f((1, ..., 1)) = 1 = f(2^v) = f((2^{v_1}, ..., 2^{v_n})) = 2^{vA} = 2^{(0,...,0)}$. Since $(1, ..., 1)$ and 2^v are distinct, we are done.

NOTE: Please feel free to e-mail comments, questions, and/or corrections.